AUTOMORPHISMS OF BINARY QUADRATIC FORMS

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Abstract. We compute all the automorphisms in $\mathrm{GL}_2(\mathbb{Z})$ of an integral binary quadratic form. Two tables at the end summarize the results. This note is part of [1].

1. Setup

We consider

$$Q = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here Q is a (weakly) reduced integral quadratic form, that is $x, z \neq 0$, $2|y| \leq x \leq z$, $2y, x, z \in \mathbb{Z}$ and $\det(Q) > 0$, and $M \in \mathrm{GL}_2(\mathbb{Z})$. We are looking for the couples (Q, M) such that

$$Q = M^t Q M.$$

Note first that if we replace M by $-M$, we get the same result. Therefore we only consider matrices up to multiplication by ± 1. The computation gives

$$0 = M^t Q M - Q = \begin{pmatrix} a^2x + 2acy + c^2z - x & abx + (ad+bc)y + cdz - y \\ abx + (ad+bc)y + cdz - y & b^2x + 2bdy + d^2z - z \end{pmatrix}. \tag{1.1}$$

We consider the first entry. Using the identity $u^2 + v^2 \geq 2|uv|$, we have

$$0 = a^2x + 2acy + c^2z - x \geq 2|ac|(\sqrt{xz} - |y|) - x \geq |ac|x - x.$$

Therefore we have $|ac| \leq 1$. We have to work a bit more for the last entry. Suppose that $|d| \geq 2$. Then $d^2 - 1 \geq \frac{3}{4}d^2$ and so

$$0 = b^2x + 2bdy + (d^2-1)z \geq 2|b|\sqrt{d^2-1}\sqrt{xz} - 2|bdy| \geq 2|bd|(\sqrt{3/4}\sqrt{xz} - |y|).$$

Since $\sqrt{3/4} > 1/2$, we have $b = 0$. Therefore we have two cases: $|d| \leq 1$ or $b = 0$.

2. Diagonal and antidiagonal M

We begin with the two easy cases of diagonal and antidiagonal matrix M. There are 4 possibilities up to multiplication by -1:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The identity is an automorphism for any matrix Q. Looking at Equation (1.1), we get respectively for the other three matrices

$$0 = \begin{pmatrix} 0 & -2y \\ -2y & 0 \end{pmatrix}, \begin{pmatrix} z-x & -2y \\ -2y & x-z \end{pmatrix}, \begin{pmatrix} x-z & 0 \\ 0 & z-x \end{pmatrix}.$$

Therefore the conditions on Q are respectively $y = 0$, $x = z \wedge y = 0$ and $x = z$.

3. Diagonal Q

We quickly consider the case $y = 0$, so we can rule out this later. Equation (1.1) rewrites as

$$0 = \begin{pmatrix} a^2x + c^2z - x & abx + cdz \\ abx + cdz & b^2x + d^2z - z \end{pmatrix}.$$

First, if $a = 0$, then $b, c = \pm 1$ since the determinant is $bc = \pm 1$. The first entry gives $x = z$ and the second entry gives $d = 0$. If $c = 0$, then $a, d = \pm 1$ and the diagonal entries vanish. The second entry gives $b = 0$. In both cases, we are back to a diagonal or antidiagonal M. Otherwise, if $ac = \pm 1$, then the first entry gives $z = 0$ which is a contradiction. So all these cases fit in the last section. From now, we suppose that $y \neq 0$

4. The case $ac = 0$

If $c = 0$, then automatically a and d equal ± 1 since the determinant is ad. That gives the matrices

$$M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & -1 \end{pmatrix}$$

for n a non-zero integer. The other cases can be obtained by multiplying by -1. Looking at Equation (1.1), we have

$$0 = \begin{pmatrix} 0 & anx + (ad-1)y \\ anx + (ad-1)y & n^2x + 2dny \end{pmatrix}.$$

So if $ad = 1$ like in the first case, then $x = 0$ and there is no such Q. In the second case, $ad = -1$ and we get $nx = 2y$ or $nx + 2y = 0$. Since $x \geq 2|y|$, we get $n = \operatorname{sgn}(y)$ and $x = 2|y|$. Now, if $a = 0$ then $bc = \pm 1$ and we have the matrices

$$M = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}.$$

Equation (1.1) rewrite as

$$0 = \begin{pmatrix} z - x & (bc-1)y + cnz \\ (bc-1)y + cnz & x + 2bny + (n^2-1)z \end{pmatrix}.$$

If $bc = 1$, then $z = 0$ and there is no such matrix. Otherwise, $x = z$ and we get the two equations $nx = 2y$ and $nx + 2y = 0$. Again, $x \geq 2|y|$ so $n = -\operatorname{sgn}(y)$ and $x = 2|y|$.

5. The case $ac = 1$

We have $a = c = \pm 1$, without loss of generality say $a = c = 1$. Therefore the first entry of the matrix is $2y + z = 0$. Since $2|y| \leq x \leq z$, we get $-2y = x = z$. Equation (1.1) rewrites as

$$0 = \begin{pmatrix} 0 & -by - dy - y \\ -by - dy - y & -2b^2y + 2bdy - 2(d^2-1)y \end{pmatrix}.$$

If $b = 0$, then the second entry gives $d + 1 = 0$ so $d = -1$ and this is compatible with the last entry. If $b \neq 0$, then we have two cases. If $d = 0$, then the second equation gives $b = -1$. This is compatible with the last entry. If $d = \pm 1$, then the last entry is $-2b^2y + 2bdy = 0$, so that $b = d$. There is no such matrix with determinant ± 1 and it is also incompatible with the second entry.

6. The case $ac = -1$

We have $a = -c = \pm 1$, without loss of generality say $a = -c = 1$. So the first entry of Equation (1.1) gives $2y = z$. Since $2|y| \leq x \leq z$, we have $2y = x = z$. The full matrix rewrites

$$0 = \begin{pmatrix} 0 & by - dy - y \\ by - dy - y & 2b^2y + 2bdy + 2d^2y - 2y \end{pmatrix}.$$

If $b = 0$, then the second entry gives $d = -1$ and is compatible with the last. If $b \neq 0$, then $d = 0$ gives $b = 1$ for both equations. If $d = \pm 1$, then the last entry is $2b^2y + 2bdy = 0$ so $b = -d$. This is incompatible with the second entry that says $b = d + 1$ (for integral b and d).

7. Summary

We summarize the result in the table below. The first column indicates the sign of the determinant of M. For each matrix M, there is the matrix $-M$ that has the same action on Q. Note that except for the fourth entry, y is always supposed to be non-zero.

$\det(M)$	M	Q
$+$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Any
$-$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$
$+$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$
$-$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} x & y \\ y & x \end{pmatrix}$
$-$	$\begin{pmatrix} 1 & \pm 1 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 2y & \pm y \\ \pm y & z \end{pmatrix}$
$+$	$\begin{pmatrix} 0 & 1 \\ -1 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} 2y & \pm y \\ \pm y & 2y \end{pmatrix}$
$-$	$\begin{pmatrix} 1 & 0 \\ \pm 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 2y & \mp y \\ \mp y & 2y \end{pmatrix}$
$+$	$\begin{pmatrix} \pm 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2y & \pm y \\ \pm y & 2y \end{pmatrix}$

We rewrite this table in terms of Q. The second column lists all the automorphisms of Q (modulo $\pm id$). The three following columns indicates respectively the number of automorphisms in $\mathrm{SL}_2(\mathbb{Z})$, in $\mathrm{GL}_2(\mathbb{Z})$ and the ratio between the two. The last column gives the corresponding Heegner point $z = \frac{-y+i\sqrt{xz-y^2}}{x}$. Here $y \neq 0$ everywhere and $y > 0$ except in the third row. We say that Q is reduced if $x = z$ or $x = 2|y|$. In that case we can, furthermore, suppose that $y > 0$. This removes the fifth and the seventh rows.

Q	M	$\mathrm{SL}_2(\mathbb{Z})$	$\mathrm{GL}_2(\mathbb{Z})$	Ratio	Heegner pt
$\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	2	4	2	$i\sqrt{\frac{z}{x}}$
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	4	8	2	i
$\begin{pmatrix} x & y \\ y & x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2	4	2	$\frac{-y+i\sqrt{x^2-y^2}}{x}$
$\begin{pmatrix} 2y & y \\ y & z \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	2	4	2	$\frac{-1}{2} + i\frac{\sqrt{2z-y}}{2\sqrt{y}}$
$\begin{pmatrix} 2y & -y \\ -y & z \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	2	4	2	$\frac{1}{2} + i\frac{\sqrt{2z-y}}{2\sqrt{y}}$
$\begin{pmatrix} 2y & y \\ y & 2y \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	12	2	$\frac{-1+i\sqrt{3}}{2}$
$\begin{pmatrix} 2y & -y \\ -y & 2y \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$	6	12	2	$\frac{1+i\sqrt{3}}{2}$
Other	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	2	1	$\frac{-y+i\sqrt{xz-y^2}}{x}$

References

[1] Gilles Felber. A restriction norm problem for siegel modular forms. August 2023.

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