BOUNDS ON WHITTAKER FUNCTIONS FOR GL(n)

GILLES FELBER AND DÁVID TÓTH

Abstract. We prove a new estimate on Jacquet-Whittaker function for $\mathrm{GL}_n(\mathbb{R})$, assuming that the Langlands parameters are purely imaginary and well-spaced. This gives an upper bound for the global sup-norm of the Whittaker function that matches the lower bound of Brumley-Templier in the exponent up to a linear error in n.

1. Introduction

In this paper, we study the (standard archimedean spherical) Jacquet–Whittaker function $\mathcal{W}_\mu$ on $\mathrm{GL}_n(\mathbb{R})$ associated with a cusp form ϕ and indexed by the Langlands parameters

$$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n \quad \text{with} \quad \mu_1 + \cdots + \mu_n = 0, \tag{1}$$

so that the automorphic representation induced by ϕ is tempered, yielding that

$$\mu = (\mu_1, \dots, \mu_n) \in (i\mathbb{R})^n \tag{2}$$

holds as well. We will also assume that the parameters are well-spaced, i.e.

$$|\mu_i - \mu_j| \geq cT_\mu \qquad (1 \leq i < j \leq n) \tag{3}$$

holds for some constant $c > 0$, where

$$T_\mu := \max(2, |\mu_1|, \dots, |\mu_n|). \tag{4}$$

This covers 99% of all Maass forms (see the introduction of [BB20] for more details). The constant c will be implicit in the rest of the article and we will use the Vinogradov notation $\gg$ to indicate (3) from now on. Recall that $T_\mu \asymp_n \lambda_\phi^{1/2}$ is related to the Laplace eigenvalue of the cusp form ϕ [BHM20, Equation (7)].

The function $\mathcal{W}_\mu$ is defined as follows. By the Iwasawa decomposition, every matrix $g \in \mathrm{GL}_n(\mathbb{R})$ can be uniquely written as $g = utk$, where $u \in \mathrm{U}_n(\mathbb{R})$ is unipotent upper-triangular, $t = \mathrm{diag}(t_1, \dots, t_n)$ is diagonal with positive diagonal entries, and $k \in \mathrm{O}_n(\mathbb{R})$ is orthogonal. The height function H_μ is given for every $\mu \in \mathbb{C}^n$ by

$$H_\mu(g) = \prod_{j=1}^{n} t_j^{(n+1)/2 - j + \mu_j} \quad (g \in \mathrm{GL}_n(\mathbb{R})).$$

It is right-invariant under $\mathrm{O}_n(\mathbb{R})$ by the Iwasawa decomposition, and if in addition (1) is satisfied, then it is also invariant under $\mathrm{Z}_n(\mathbb{R})$ (the center of $\mathrm{GL}_n(\mathbb{R})$), so it can be regarded

(G. Felber and D. Tóth) HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary

(D. Tóth) Department of Algebra and Geometry, Institute of Mathematics, Budapest University of Technology and Economics, Műegyetem rkp. 3, H-1111 Budapest, Hungary

Email address: felber@renyi.hu, toth.david.akos@ttk.bme.hu.

2020 *Mathematics Subject Classification.* 11F30.

Key words and phrases. Jacquet-Whittaker function, automorphic forms on GL(n), Stade formula, sup-norm problem.

as a function on the generalized upper half-plane $\mathfrak{h}^n = \mathrm{GL}_n(\mathbb{R})/(\mathrm{O}_n(\mathbb{R}) \cdot \mathrm{Z}_n(\mathbb{R}))$. We define the Jacquet–Whittaker function by

$$\mathcal{W}_\mu(g) = \int_{\mathrm{U}_n(\mathbb{R})} H_\mu(wug)\overline{\psi(u)}\,du \tag{5}$$

where w is the long Weyl element and

$$\psi(u) = \exp(2\pi i(u_{1,2} + \cdots + u_{n-1,n})), \quad u = (u_{ij}) \in \mathrm{U}_n(\mathbb{R}).$$

The integral in (5) converges locally uniformly for any $g \in \mathrm{GL}_n(\mathbb{R})$ and for any $\mu \in \mathbb{C}^n$ such that $\mathrm{Re}\,\mu_1 > \cdots > \mathrm{Re}\,\mu_n$, and hence defines a holomorphic function $\mu \mapsto \mathcal{W}_\mu(g)$. Jacquet [Jac67] proved the holomorphic continuation of this function for any $\mu \in \mathbb{C}^n$.

All this implies that $\mathcal{W}_\mu(g)$ is invariant under $\mathrm{Z}_n(\mathbb{R})$ once (1) is satisfied, right-invariant under $\mathrm{O}_n(\mathbb{R})$ and transforms by a character under the left action of $\mathrm{U}_n(\mathbb{R})$, hence it is enough to estimate it on the positive diagonal torus. Note that Jacquet [Jac67] also proved that the completed Jacquet–Whittaker function

$$W_\mu(g) := \left(\prod_{1\le j<k\le n} \Gamma_\mathbb{R}(1+\mu_j-\mu_k)\right)\mathcal{W}_\mu(g)$$

is invariant under any permutation of the μ_j's (here $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$), hence we may (and will) assume

$$\mathrm{Im}\,\mu_1 \ge \cdots \ge \mathrm{Im}\,\mu_n$$

as well. We are going to show the following:

Theorem 1.1. *Let $n \ge 3$, $t = \mathrm{diag}(t_1, \dots, t_n) \in \mathrm{GL}_n(\mathbb{R})$ with $t_1, \dots, t_n > 0$, and assume that the Langlands parameters satisfy* (1), (2) *and* (3). *Then for any $\epsilon > 0$ we have*

$$\mathcal{W}_\mu(\mathrm{diag}(t_1,\dots,t_n)) \ll_{n,\epsilon} T_\mu^{-\frac{3n^2-11n+14}{12}+\epsilon}\left(\prod_{j=1}^n t_j^{n+1-2j}\right)^{1/2-\epsilon}\exp\left(-\frac{1}{T_\mu}\sum_{j=1}^{n-1} t_j/t_{j+1}\right). \tag{6}$$

Note that as $\mathcal{W}_\mu$ is invariant under $\mathrm{Z}_n(\mathbb{R})$, it is clearly enough to show (6) for $t = \mathrm{diag}(t_1, \dots, t_{n-1}, 1)$, or equivalently, using the parametrization of [BT20, Sta90, Sta01, Sta02], it suffices to prove

$$\mathcal{W}_\mu(\mathrm{diag}(y_1\dots y_{n-1},\dots,y_{n-1},1)) \ll_{n,\epsilon} T_\mu^{-\frac{3n^2-11n+14}{12}+\epsilon}\prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}-\epsilon}\exp\left(-\frac{1}{T_\mu}\sum_{j=1}^{n-1} y_j\right).$$

This is the estimate that we are going to prove in Section 3. Assumption (3) is only needed in Section 3.3, when we estimate the Gamma factors. A bound involving quotients of Gamma functions can be deduced assuming only that $|\mu_1 - \mu_n| \gg 1$.

As in [BHM20, Theorem 1], (6) implies

$$\|\mathcal{W}_\mu\|_\infty \ll_{n,\epsilon} T_\mu^{\frac{(n^3-n)-(3n^2-11n+14)}{12}+\epsilon} = T_\mu^{\frac{n^3-3n^2+10n-14}{12}+\epsilon}. \tag{7}$$

This complements [BT20, Theorem 1.14] that states (under the same conditions)

$$\|\mathcal{W}_\mu\|_\infty \gg_n T_\mu^{n(n-1)(n-2)/12}. \tag{8}$$

Note that the cubic and the quadratic terms of the exponent agree in the upper and lower bounds above. More precisely, the difference in the exponent between our result and [BT20] is $\frac{2}{3}(n-\frac{7}{4})$. Also, the upper bound (7) improves on the one given in [BHM20, Remark 1] if $n \ge 4$, and the main ingredient of our improvement is the use of a Mellin transform computed by Stade (see Equation (12)). For $n=3$ the two bounds are the same, and in this case the improved bound $\|\mathcal{W}_\mu\|_\infty \gg T_\mu^{3/4}$ is stated in [BT20, Theorem 1.9] while the upper estimate $\|\mathcal{W}_\mu\|_\infty \ll T_\mu$ is given in [BHM19, Lemma 2] (while we get $\|\mathcal{W}_\mu\|_\infty \ll T_\mu^{4/3}$). We expect the optimal exponent for $n \geq 4$ to sit strictly between the two bounds (7) and (8).

One of the main interest of global bounds on Whittaker functions is the global sup-norm problem for cusp forms on $\mathrm{GL}_n(\mathbb{R})$. The best known lower bounds for this problem are given by [BT20] for $n \geq 3$ and the best known upper bounds can be found in [BHM19] for $n=3$ and [BHM20] for $n \geq 4$. See also the citations therein for a more detailed account of the literature. Following the proof of [BHM20], we can give a modest improvement on the bound of [BHM20, Theorem 4] for the global sup-norm problem for cusp forms on $\mathrm{GL}_n(\mathbb{R})$. The improvement is only in the quadratic term of the exponent. It would be of great interest to lower their bound in the cubic term and match the corresponding term of the lower bound. One approach would be to refine the counting argument in Section 3.2 of [BHM20]. Another interesting question is the case of non-tempered automorphic representations. In that case, we know that the real parts of the components of μ are strictly between $-\frac{1}{2}$ and $\frac{1}{2}$ (see [BHM20, Equation (5)]). It would be of interest to improve the bound of [BHM20, Theorem 2] in that generality. We hope to return to these problems soon.

1.1. **Acknowledgment.** The authors are profoundly thankful to Péter Maga, who introduced them to this topic and guided them in this project. The research towards this paper was supported by the MTA–RI Lendület "Momentum" Analytic Number Theory and Representation Theory Research Group for both authors.

2. Bounds on integrals of K-Bessel functions

In this section we prove an estimate for the K-Bessel functions that will be used frequently in the proof of Theorem 1.1. In the course of the proof we are going to the need the following:

Proposition 2.1. *Let $a > 0$, $b \in \mathbb{R}$ and $x > 0$. Assume that $b \leq 1$ or $ax \gg 1$, then we have*

$$\int_x^\infty e^{-at} t^b \frac{dt}{t} \ll_b e^{-ax} a^{-1} x^{b-1}. \tag{9}$$

Proof. If $b \leq 1$, then

$$\int_x^\infty e^{-at} t^b \frac{dt}{t} \leq x^{b-1} \int_x^\infty e^{-at} dt \leq x^{b-1} a^{-1} e^{-ax}.$$

If $b > 1$ and $ax \gg 1$, then the statement can be proved by induction using integration by parts. □

Lemma 2.2. *Let $r \gg 1$, $y > 0$ and $\ell, k > 0$ fixed. Then we have*

$$\int_0^\infty \left| K_{ir}(2\pi y \sqrt{1+u^{\pm 2}}) \right|^\ell u^{\pm k} \frac{du}{u} \ll_{\ell,k} e^{-\pi \ell r/2} r^{k-\ell/3} y^{-k}.$$

Moreover, if $2\pi y > 1 + \pi r/2$, then

$$\int_0^\infty \left| K_{ir}(2\pi y \sqrt{1+u^{\pm 2}}) \right|^\ell u^{\pm k} \frac{du}{u} \ll_{\ell,k} e^{-\pi \ell y} y^{-k-\ell/2}.$$

If in addition $r \gg T_\mu$ holds, where T_μ is defined in (4), then we also have the uniform estimate

$$\int_0^\infty \left| K_{ir}(2\pi y \sqrt{1+u^{\pm 2}}) \right|^\ell u^{\pm k} \frac{du}{u} \ll_{\ell,k} e^{-\ell y/T_\mu} e^{-\pi \ell r/2} r^{k-\ell/3} y^{-k}.$$

Proof. We use the following bounds for the K-Bessel function of imaginary argument: if $x>0$, $r\geq 1$, then we have (see [BH09, p. 679])

$$e^{\pi r/2}\left|K_{ir}(x)\right| \ll \min(r^{-1/3}, \left|r^2-x^2\right|^{-1/4}),$$

and so

$$K_{ir}(x) \ll e^{-\pi r/2} r^{-1/3} \tag{10}$$

holds for every $x>0$ and $r\geq 1$. On the other hand, if $x>1+\pi r/2$, then the estimate

$$|K_{ir}(x)| \ll e^{-x} x^{-1/2} \tag{11}$$

holds by [HM05, Proposition 9] for all $r\in\mathbb{R}$.

If $x>0$ but $0<\delta<r<1$, then the function $K_{ir}(x)$ is uniformly bounded by a constant only depending on δ. We show this by splitting the usual integral representation for $K_{ir}(x)$:

$$\begin{aligned}
K_{ir}(x) &= \int_0^\infty e^{-x\cosh(t)}\cos(rt)dt \\
&= \int_0^1 e^{-x\cosh(t)}\cos(rt)dt + \int_1^\infty e^{-x\cosh(t)}\cos(rt)dt \\
&\ll 1 + e^{-x\cosh(t)}\frac{\sin(rt)}{r}\bigg|_1^\infty + \frac{x}{r}\int_1^\infty e^{-x\cosh(t)}\sinh(t)\sin(rt)dt \\
&\ll 1 + \frac{1}{r} + \frac{x}{r}\int_1^\infty e^{-x\cosh(t)}\sinh(t)dt \\
&\ll 1 + \frac{1}{r} - \frac{x}{r}\frac{e^{-x\cosh(t)}}{x}\bigg|_1^\infty \\
&\ll 1 + \frac{1}{r} + \frac{1}{r}.
\end{aligned}$$

This shows that Equation (10) is valid for all $x>0$ and $r\gg 1$ (where the implied constant depends on δ).

Turning to the proof of the statements, first note that it is enough to prove them with positive signs in the exponents and then the change of variables $u\mapsto u^{-1}$ gives the result with negative signs.

For the first bound we split the integral at the point $u=\frac{\pi r+2}{4\pi y}$:

$$\begin{aligned}
&\int_0^\infty \left|K_{ir}(2\pi y\sqrt{1+u^2})\right|^\ell u^k\frac{du}{u} = \\
&\quad = \int_0^{\frac{\pi r+2}{4\pi y}} \left|K_{ir}(2\pi y\sqrt{1+u^2})\right|^\ell u^k\frac{du}{u} + \int_{\frac{\pi r+2}{4\pi y}}^\infty \left|K_{ir}(2\pi y\sqrt{1+u^2})\right|^\ell u^k\frac{du}{u} =: I_1+I_2.
\end{aligned}$$

In the case of I_1, we apply the bound (10) yielding

$$\int_0^{\frac{\pi r+2}{4\pi y}} \left|K_{ir}(2\pi y\sqrt{1+u^2})\right|^\ell u^k\frac{du}{u} \ll_\ell e^{-\pi\ell r/2}r^{-\ell/3}\int_0^{\frac{\pi r+2}{4\pi y}} u^k\frac{du}{u} \ll_{\ell,k} e^{-\pi\ell r/2}r^{k-\ell/3}y^{-k}.$$

Now consider I_2. In that case, we have $2\pi y\sqrt{1+u^2} \geq \frac{\pi r+2}{2}$. Using first (11) and then (9), we get:

$$I_2 \ll_\ell \int_{\frac{\pi r+2}{4\pi y}}^\infty e^{-2\pi\ell y\sqrt{1+u^2}} \left(y\sqrt{1+u^2}\right)^{-\ell/2} u^k \frac{du}{u}$$
$$\ll_\ell y^{-\ell/2} \int_{\frac{\pi r+2}{4\pi y}}^\infty e^{-2\pi\ell yu} u^{k-\ell/2} \frac{du}{u}$$
$$\ll_{\ell,k} e^{-\pi\ell r/2} y^{-\ell/2} y^{-1} (r/y)^{k-\ell/2-1} = e^{-\pi\ell r/2} r^{k-\ell/2-1} y^{-k}.$$

If $y \geq \frac{\pi r+2}{4\pi}$, then using $\sqrt{1+u^2} \geq \frac{1+u}{2}$ the corresponding bound is an easy consequence of (11):

$$\int_0^\infty \left|K_{ir}(2\pi y\sqrt{1+u^2})\right|^\ell u^k \frac{du}{u} \ll_\ell \int_0^\infty e^{-\pi\ell y(1+u)} y^{-\ell/2} u^k \frac{du}{u}$$
$$\ll_{\ell,k} e^{-\pi\ell y} y^{-k-\ell/2} \int_0^\infty e^{-t} t^k \frac{dt}{t}$$
$$\ll_{\ell,k} e^{-\pi\ell y} y^{-k-\ell/2}.$$

We conclude with the proof of the last bound, so assume that $r \gg T_\mu$ holds. Since $e^{-\ell y/r}$ is bounded from below for $y \leq 1+r$, so is $e^{-\ell y/T_\mu}$. Hence the first bound implies the third one. If $y > 1+r$, then $2\pi y > 1 + \pi r/2$ and $y^{-\ell/2} \ll r^{-\ell/3}$. Now the third bound follows from the second one:

$$\int_0^\infty \left|K_{ir}(2\pi y\sqrt{1+u^{\pm 2}})\right|^\ell u^k \frac{du}{u} \ll_{\ell,k} e^{-\pi\ell y} y^{-k-\ell/2}$$
$$\ll_{\ell,k} e^{-\pi\ell y/2} e^{-\pi\ell r/2} r^k y^{-k} r^{-\ell/3}$$
$$\ll_{\ell,k} e^{-\ell y/T_\mu} e^{-\pi\ell r/2} r^{k-\ell/3} y^{-k},$$

since $T_\mu \geq 2$ by definition and $r \gg 1$. □

3. Proof of the main theorem

Before turning to the argument we state the following formula of Stade that will serve as one of our tools (see [BHM20, Equation (33)], [Sta02, Theorem 1.1]). Using the notations of Section 1, for an appropriate $s \in \mathbb{C}$ we have

$$\int_{\mathbb{R}_{>0}^{n-1}} |W_\mu(\mathrm{diag}(t_1, \ldots, t_{n-1}, 1))|^2 \prod_{j=1}^{n-1} \frac{t_j^{s-1}\, dt_j}{t_j^{n+1-2j}} = \frac{2^{1-n}}{\Gamma_\mathbb{R}(ns)} \prod_{k,l=1}^n \Gamma_\mathbb{R}(s + \mu_k - \mu_l). \tag{12}$$

To handle the convergence of the integral on the left-hand side we change to y-notation, i.e., make the change of variables $y_j \cdots y_{n-1} = t_j$ ($1 \leq j \leq n-1$). The determinant of the (upper-triangular) Jacobian is $\prod_{j=1}^{n-1} y_j^{j-1}$, and left-hand side of (12) is

$$\int_{\mathbb{R}_{>0}^{n-1}} |W_\mu(\mathrm{diag}(y_1 \cdots y_{n-1}, \ldots, y_{n-1}, 1))|^2 \prod_{j=1}^{n-1} y_j^{j(s+j-n)} \frac{dy_j}{y_j}.$$

By [BHM20, Equation (14)], we have

$$|W_\mu(\mathrm{diag}(y_1 \ldots y_{n-1}, \ldots, y_{n-1}, 1))| \ll_{n,\mu,\epsilon} \left(\prod_{j=1}^{n-1} y_j^{j(n-j)}\right)^{1/2-\epsilon} \exp\left(-\frac{1}{T_\mu}\sum_{j=1}^{n-1} y_j\right)$$

for any $\epsilon > 0$, where $T_\mu = \max(2, |\mu_1|, \ldots, |\mu_n|)$. Hence the formula holds if the integral of

$$\prod_{j=1}^{n-1} y_j^{j(n-j)(1-\epsilon)+j(s+j-n)-1} = \prod_{j=1}^{n-1} y_j^{js-1-\epsilon j(n-j)}$$

converges in a neighborhood of 0. Choosing an appropriate $\epsilon > 0$ we get that (12) holds if $j\mathrm{Re}\, s - 1 > -1$, i.e., when $\mathrm{Re}\, s > 0$.

The starting point of our proof is a recursion formula of Stade. Before stating it we introduce the notation

$$W_\mu^*(y_1, \ldots, y_{n-1}) = \\ W_\mu(\mathrm{diag}(y_1 \ldots y_{n-1}, y_{n-2} \ldots y_{n-1}, \ldots, y_{n-1}, 1)) \prod_{i=1}^{n-1} y_i^{-\frac{i(n-i)}{2}} \prod_{j=1}^{i} y_i^{-\frac{\mu_j + \mu_{n+1-j}}{2}},$$

then we have the following (see [BHM20, Equation (32)], [Sta01, Equation (4.3)]):

Theorem 3.1. *If $n \geq 2$, then*

$$W_\mu^*(y_1, \ldots, y_{n-1}) = 2^{2n-3} \int_{\mathbb{R}_{>0}^{n-2}} \prod_{j=1}^{n-1} u_j^{\frac{\mu_{j+1}+\mu_{n-j}-\mu_1-\mu_n}{2}} K_{\frac{\mu_1-\mu_n}{2}} \left(2\pi y_j \sqrt{(1+u_{j-1}^2)(1+u_j^{-2})}\right) \\ \cdot W_{\mu'}^* \left(y_2 \frac{u_1}{u_2}, \ldots, y_{n-2} \frac{u_{n-3}}{u_{n-2}}\right) \frac{du_1 \cdots du_{n-2}}{u_1 \cdots u_{n-2}}$$

where

$$\mu' = \left(\mu_2 + \frac{\mu_1 + \mu_n}{n-2}, \ldots, \mu_{n-1} + \frac{\mu_1 + \mu_n}{n-2}\right) \in \mathbb{C}^{n-2}$$

$u_0 = u_{n-1}^{-1} = 0$, $u_{n-1}^0 = 1$, *and for $n = 2, 3$ the inner Whittaker function $W_{\mu'}^*$ is understood to be equal* 1.

The outline of the proof is the following: we apply the Cauchy-Schwarz inequality to separate the inner Whittaker function on the right-hand side of the above formula. To ensure convergence we also include one Bessel function and a product $P(u_1, \ldots, u_{n-2})$ of powers of u_j next to it so that $2^{-4n+6}|W_\mu^*(y_1, \ldots, y_{n-1})|^2$ is bounded by

$$\left(\int_{\mathbb{R}_{>0}^{n-2}} \prod_{j=1}^{n-2} |K_j|^2 \, |K_{n-1}| \, P(u_1, \ldots, u_{n-2})^{-1}\right) \left(\int_{\mathbb{R}_{>0}^{n-2}} \left|W_{\mu'}^{*(n-2)}\right|^2 |K_{n-1}| \, P(u_1, \ldots, u_{n-2})\right)$$

(written briefly) where we also use that $\mathrm{Re}\, \mu_j = 0$ for $1 \leq j \leq n$. Here and later we indicate the rank in the notation of the Whittaker functions. Note that the factor

$$K_{n-1} = K_{\frac{\mu_1-\mu_n}{2}} \left(2\pi y_{n-1} \sqrt{1 + u_{n-2}^2}\right)$$

only depends on u_{n-2} and decays exponentially as $u_{n-2} \to \infty$. For the right factor of the previous product we will apply (12), while in the case of the left factor we will use the Cauchy-Schwarz inequality and induction on n. We will use the same power for each u_j, that is, we set $P(u_1, \ldots, u_{n-2}) = \prod_{j=1}^{n-2} u_j^s$.

From now on, we write $r = \mathrm{Im}\left(\frac{\mu_1-\mu_n}{2}\right)$. We have $r \gg 1$ where we emphasize that the implied constant depends directly on the condition of Equation (3).

3.1. **Estimation of the right factor.** Let us set

$$I_R := \int_{\mathbb{R}_{>0}^{n-2}} \left| W_{\mu'}^{*(n-2)} \left(y_2 \frac{u_1}{u_2}, \ldots, y_{n-2} \frac{u_{n-3}}{u_{n-2}} \right) \right|^2 \left| K_{ir} \left(2\pi y_{n-1} \sqrt{1+u_{n-2}^2} \right) \right| \prod_{j=1}^{n-2} u_j^s \frac{du_j}{u_j}$$

We are going to show the following:

Proposition 3.2. *If $s > 0$, then*

$$I_R \ll_{n,s} \prod_{j=1}^{n-1} y_j^{-(j-1)s} \prod_{k,l=2}^{n-1} \Gamma_{\mathbb{R}}(s + \mu_k - \mu_l) e^{-\pi r/2 - y_{n-1}/T_\mu} r^{(n-2)s-1/3}.$$

For the proof we first change to the function $W_{\mu'}^{(n-2)}$ from $W_{\mu'}^{*(n-2)}$ and also make a change of variables, i.e., we set

$$t_j = \prod_{k=j}^{n-3} y_{k+1} \frac{u_k}{u_{k+1}} = y_{j+1} \cdots y_{n-2} \frac{u_j}{u_{n-2}}$$

for $j = 1, \ldots, n-2$ with the convention $t_{n-2} = 1$ (meaning that the value of the empty product is 1). We also have

$$u_j = \frac{t_j u_{n-2}}{y_{j+1} \cdots y_{n-2}} \qquad (1 \le j \le n-2).$$

Note that we do not apply a change of variable to u_{n-2}. Then I_R can be written as

$$\begin{aligned} &\prod_{j=1}^{n-3} (y_{j+1} \cdots y_{n-2})^{-s} \int_{\mathbb{R}_{>0}^{n-2}} \left| W_{\mu'}^{(n-2)} (\mathrm{diag}(t_1, \ldots, t_{n-3}, 1)) \right|^2 \left| K_{ir} \left(2\pi y_{n-1} \sqrt{1+u_{n-2}^2} \right) \right| \\ &\qquad \times \prod_{j=1}^{n-3} \left[\left(\frac{t_j}{t_{j+1}} \right)^{-j(n-2-j)} (t_j u_{n-2})^s \frac{dt_j}{t_j} \right] u_{n-2}^s \frac{du_{n-2}}{u_{n-2}} \\ &= \prod_{j=2}^{n-2} y_j^{-(j-1)s} \int_{\mathbb{R}_{>0}^{n-3}} \left| W_{\mu'}^{(n-2)} (\mathrm{diag}(t_1, \ldots, t_{n-3}, 1)) \right|^2 \prod_{j=1}^{n-3} t_j^{s+2j-n+1} \frac{dt_j}{t_j} \\ &\qquad \times \int_{\mathbb{R}_{>0}} \left| K_{ir} \left(2\pi y_{n-1} \sqrt{1+u_{n-2}^2} \right) \right| u_{n-2}^{(n-2)s} \frac{du_{n-2}}{u_{n-2}}. \end{aligned}$$

Setting $Y := \prod_{j=2}^{n-2} y_j^{-(j-1)s}$ and applying (12) for the first integral (with $W_{\mu'}^{(n-2)}$ in place of $W_\mu^{(n)}$) we obtain

$$I_R = \frac{2^{3-n} \cdot Y}{\Gamma_{\mathbb{R}}((n-2)s)} \prod_{k,l=2}^{n-1} \Gamma_{\mathbb{R}}(s + \mu_k - \mu_l) \int_0^\infty \left| K_{ir} \left(2\pi y_{n-1} \sqrt{1+u_{n-2}^2} \right) \right| u_{n-2}^{(n-2)s} \frac{du_{n-2}}{u_{n-2}}.$$

Proposition 3.2 follows now from the third estimate of Lemma 2.2.

3.2. **Estimation of the left factor.** Let us set

$$I_L = \int_{\mathbb{R}_{>0}^{n-2}} \prod_{j=1}^{n-2} K_{ir} \left(2\pi y_j \sqrt{(1+u_{j-1}^2)(1+u_j^{-2})} \right)^2 \left| K_{ir} \left(2\pi y_{n-1} \sqrt{1+u_{n-2}^2} \right) \right| \prod_{j=1}^{n-2} u_j^{-s} \frac{du_j}{u_j}.$$

First we isolate the integral over u_{n-2} and apply the Cauchy-Schwarz inequality:

$$\int_0^\infty K_{ir}\left(2\pi y_{n-2}\sqrt{(1+u_{n-3}^2)(1+u_{n-2}^{-2})}\right)^2\left|K_{ir}\left(2\pi y_{n-1}\sqrt{1+u_{n-2}^2}\right)\right|u_{n-2}^{-s}\frac{du_{n-2}}{u_{n-2}}$$
$$\ll\left(\int_0^\infty K_{ir}\left(2\pi y_{n-2}\sqrt{(1+u_{n-3}^2)(1+u_{n-2}^{-2})}\right)^4u_{n-2}^{-2s-\gamma}\frac{du_{n-2}}{u_{n-2}}\right)^{1/2}$$
$$\times\left(\int_0^\infty K_{ir}\left(2\pi y_{n-1}\sqrt{1+u_{n-2}^2}\right)^2u_{n-2}^{\gamma}\frac{du_{n-2}}{u_{n-2}}\right)^{1/2}.$$

We assume that $\gamma>0$ and keep our previous assumption $s>0$ so that $2s+\gamma>0$ holds, and apply Lemma 2.2 on the right-hand side: in the case of the first factor we have the parameters $\ell=4$, $k=2s+\gamma$, $y=y_{n-2}\sqrt{1+u_{n-3}^2}$, while in the case of the second factor we have $\ell=2$, $k=\gamma$, $y=y_{n-1}$. Then the integral over u_{n-2} is bounded by

$$e^{-\pi r-\frac{2y_{n-2}\sqrt{1+u_{n-3}^2}}{T_\mu}}r^{s+\gamma/2-2/3}\left(y_{n-2}\sqrt{1+u_{n-3}^2}\right)^{-s-\gamma/2}\cdot e^{-\frac{\pi r}{2}-\frac{y_{n-1}}{T_\mu}}r^{\gamma/2-1/3}y_{n-1}^{-\gamma/2}=$$

$$=\exp\left(-\frac{3\pi r}{2}-\frac{2y_{n-2}\sqrt{1+u_{n-3}^2}}{T_\mu}-\frac{y_{n-1}}{T_\mu}\right)r^{s+\gamma-1}y_{n-2}^{-s-\gamma/2}y_{n-1}^{-\gamma/2}(1+u_{n-3}^2)^{-\frac{2s+\gamma}{4}}.$$

We estimate this further by estimating $1+u_{n-3}^2$ from below. We use $1+u_{n-3}^2\geq 1$ in the exponential while in the case $n>3$ estimate the last factor by $1+u_{n-3}^2\geq u_{n-3}^2$ to obtain the bound $B_{n-2}u_{n-3}^{-s-\gamma/2}$, where

$$B_{n-2}:=\exp\left(-\frac{3\pi r}{2}-\frac{2y_{n-2}+y_{n-1}}{T_\mu}\right)r^{s+\gamma-1}y_{n-2}^{-s-\gamma/2}y_{n-1}^{-\gamma/2}. \tag{13}$$

Let us summarize this in the following estimates: if $n>3$, then

$$I_L\ll_{s,\gamma}B_{n-2}\int_{\mathbb{R}_{>0}^{n-3}}\prod_{j=1}^{n-3}K_{ir}\left(2\pi y_j\sqrt{(1+u_{j-1}^2)(1+u_j^{-2})}\right)^2\left(\prod_{j=1}^{n-4}u_j^{-s}\frac{du_i}{u_i}\right)u_{n-3}^{-2s-\frac{\gamma}{2}}\frac{du_{n-3}}{u_{n-3}}. \tag{14}$$

Note that in the case $n=3$ our bound for I_L is simply B_{n-2}. We now give an upper bound for the latter integral by induction.

Lemma 3.3. *Let $n\in\mathbb{N}^+$, $r\geq 1$ and $y_1,\dots,y_n>0$. If $\sum_{m=j}^n k_m$ is positive, then*

$$\int_{\mathbb{R}_{>0}^n}K_{ir}\left(2\pi y_1\sqrt{1+u_1^{-2}}\right)^2\prod_{j=2}^nK_{ir}\left(2\pi y_j\sqrt{(1+u_{j-1}^2)(1+u_j^{-2})}\right)^2\prod_{j=1}^nu_j^{-k_j}\frac{du_j}{u_j}$$
$$\ll_{k_1,\dots,k_n}\exp\left(-\pi nr-\frac{2}{T_\mu}\sum_{j=1}^ny_j\right)r^{\sum_{j=1}^njk_j-2n/3}\prod_{j=1}^ny_j^{-\sum_{m=j}^nk_m}.$$

Proof. We prove by induction on n. If $n=1$, then Lemma 2.2 gives the result. Assume now that $n>1$ and the statements hold for $n-1$. We consider the integral over u_n. Again by Lemma 2.2, we have

$$\int_0^\infty K_{ir}\left(2\pi y_n\sqrt{(1+u_{n-1}^2)(1+u_n^{-2})}\right)^2u_n^{-k_n}\frac{du_n}{u_n}\ll_{k_n}e^{-\pi r-\frac{2y_n}{T_\mu}}r^{k_n-\frac{2}{3}}y_n^{-k_n}(1+u_{n-1}^2)^{-\frac{k_n}{2}}$$

since by assumption $k_n > 0$.

Now we use $(1+u_{n-1}^2)^{-k_n/2} \leq u_{n-1}^{-k_n}$, and it remains to estimate the integral

$$I_{n-1} = \int_{\mathbb{R}_{>0}^{n-1}} K_{ir}\left(2\pi y_1\sqrt{1+u_i^{-2}}\right)^2 \prod_{j=2}^{n-1} K_{ir}\left(2\pi y_i\sqrt{(1+u_{j-1}^2)(1+u_j^{-2})}\right)^2 \prod_{j=1}^{n-1} u_j^{-k_j'}\frac{du_j}{u_j},$$

where $k_j' = k_j$ for $j = 1, \ldots, n-2$, while $k_{n-1}' = k_{n-1} + k_n$. By our original assumption we have $\sum_{m=j}^{n-1} k_m' > 0$ for every $1 \leq j \leq n-1$, so the induction hypothesis gives

$$I_{n-1} \ll_{k_1,\ldots,k_{n-1}} e^{-\pi(n-1)r - \frac{2}{T_\mu}\sum_{j=1}^{n-1} y_j} r^{\sum_{j=1}^{n-1} jk_j + (n-1)k_n - 2(n-1)/3} \prod_{j=1}^{n-1} y_j^{-\sum_{m=j}^{n-1} k_m - k_n}$$

and the statement follows. $\square$

Since our earlier assumptions imply $js + \gamma/2 > 0$ for $2 \leq j \leq n-2$, we can apply the previous lemma in (14) to get

(15)

$$I_L \ll_{n,s,\gamma} B_{n-2} \cdot \exp\left(-\pi(n-3)r - \frac{2}{T_\mu}\sum_{j=1}^{n-3} y_j\right) r^{\frac{n(n-3)}{2}s + \frac{n-3}{2}\gamma - \frac{2(n-3)}{3}} \prod_{j=1}^{n-3} y_j^{-(n-1-j)s - \frac{\gamma}{2}}.$$

3.3. **Conclusion of the proof.** Now we combine the results of the previous sections to obtain the bound on the Whittaker function. Recall that $\left|W_\mu^{*(n)}(y_1, \ldots, y_{n-1})\right|^2 \ll_n I_L \cdot I_R$, hence Proposition 3.2 together with (13) and (15) imply the following: if $s, \gamma > 0$, then

$$W_\mu^{*(n)}(y_1, \ldots, y_{n-1}) \ll_{n,s,\gamma} \exp\left(-\frac{\pi(n-1)r}{2} - \frac{1}{T_\mu}\sum_{j=1}^{n-1} y_j\right) \prod_{k,l=2}^{n-1} \Gamma_{\mathbb{R}}(s + \mu_k - \mu_l)^{1/2}$$
$$\times r^{\frac{(n+1)(n-2)}{4}s + \frac{n-1}{4}\gamma - \frac{n-1}{3}} \prod_{j=1}^{n-1} y_j^{-\frac{(n-2)s}{2} - \frac{\gamma}{4}}.$$

Finally, we write $\mathcal{W}_\mu^{(n)}(\mathrm{diag}(y_1 \ldots y_{n-1}, \ldots, y_{n-1}, 1))$ as

$$W_\mu^{*(n)}(y_1, \ldots, y_{n-1}) \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}} \prod_{k=1}^{j} y_j^{\frac{\mu_k + \mu_{n+1-k}}{2}} \prod_{1 \leq k < l \leq n} \Gamma_{\mathbb{R}}(1 + \mu_k - \mu_l)^{-1}.$$

We consider first the Gamma factors. By Stirling's formula, we have

$$|\Gamma_{\mathbb{R}}(\sigma + it)| \sim_\sigma \sqrt{2\pi} e^{-\pi|t|/4} |t|^{\frac{\sigma-1}{2}}$$

for $t \to \pm\infty$. Then

$$e^{-\pi(n-1)r/2} \prod_{k,l=2}^{n-1} \Gamma_{\mathbb{R}}(s + \mu_k - \mu_l)^{1/2} \prod_{1 \leq k < l \leq n} \Gamma_{\mathbb{R}}(1 + \mu_k - \mu_l)^{-1}$$

$$\ll_{n,s} \left(\prod_{k=1}^{n-1} e^{\pi(\mu_k - \mu_n)/4}\right)\left(\prod_{l=2}^{n-1} e^{\pi(\mu_1 - \mu_l)/4}\right)\left(\prod_{\substack{k,l=2 \\ j \neq k}}^{n-1} |\mu_k - \mu_l|^{\frac{s-1}{4}}\right) e^{-\pi(n-1)(\mu_1 - \mu_n)/4}$$

$$\ll_{n,s} \prod_{\substack{k,l=2 \\ j \neq k}} |\mu_k - \mu_l|^{\frac{s-1}{4}}.$$

Since we assume that the parameter μ is well-spaced, i.e., $|\mu_k - \mu_l| \asymp T_\mu$ for any $k \neq l$, this is bounded by $T_\mu^{(n-2)(n-3)(s-1)/4}$.

We also have $r \asymp T_\mu$ and all this yields the bound

$$e^{-\frac{1}{T_\mu}\sum_{j=1}^{n-1} y_j} \prod_{j=1}^{n-1} \left(\frac{T_\mu}{y_j}\right)^{\frac{n-2}{2}s+\frac{\gamma}{4}} T_\mu^{-\frac{3n^2-11n+14}{12}} \prod_{j=1}^{n-1} y_j^{\frac{j(n-j)}{2}}.$$

Now choosing small s and γ and taking into account the comment after Theorem 1.1 as well, the proof of Theorem 1.1 is complete.

References

[BB20] Valentin Blomer and Jack Buttcane. On the subconvexity problem for L-functions on GL(3). *Annales scientifiques de l'École normale supérieure*, 53(6):1441–1500, 2020.

[BH09] Valentin Blomer and Roman Holowinsky. Bounding sup-norms of cusp forms of large level. *Inventiones mathematicae*, 179(3):645–681, December 2009.

[BHM19] Valentin Blomer, Gergely Harcos, and Péter Maga. On the global sup-norm of GL(3) cusp forms. *Israel Journal of Mathematics*, 229(1):357–379, 2019.

[BHM20] Valentin Blomer, Gergely Harcos, and Péter Maga. Analytic properties of spherical cusp forms on GL(n). *Journal d'Analyse Mathématique*, 140(2):483–510, 2020.

[BT20] Farrell Brumley and Nicolas Templier. Large values of cusp forms on GL_n. *Selecta Mathematica*, 26(4):Paper No. 63, 71 pp., December 2020.

[HM05] Gergely Harcos and Philippe Michel. The subconvexity problem for Rankin-Selberg L-functions and equidistribution of Heegner points. ii. *Inventiones mathematicae*, 163(3):581–655, 2005.

[Jac67] Hervé Jacquet. Fonctions de Whittaker associées aux groupes de Chevalley. *Bulletin de la société mathématique de France*, 95:243–309, 1967.

[Sta90] Eric Stade. On explicit integral formulas for GL($n,\mathbb{R}$)-Whittaker functions. *Duke Mathematical Journal*, 60(2):313–362, April 1990.

[Sta01] Eric Stade. Mellin transforms of GL($n,\mathbb{R}$) Whittaker functions. *Amer. J. Math.*, 123(1):121–161, February 2001.

[Sta02] Eric Stade. Archimedean L-factors on GL(n) $\times$ GL(n) and generalized Barnes integrals. *Israel Journal of Mathematics*, 127(1):201–219, December 2002.