COMPLEX FUNCTIONS

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Abstract. These notes are a complement to the textbook [SS] for the course *complex analysis* taught at BSM in 2025. May contain typos and misunderstandings. For personal use only!

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1. Preliminaries

Book: pages 1 to 8.

1.1. **Complex differentiable functions.** A complex-valued function f is complex differentiable (or holomorphic, or complex analytic) if for any point z in its domain, the limit

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists. This limit is denoted $f'(z)$. Note that the limit is taken over all *complex* numbers h. It is a much stronger requirement than real differentiation and has far more consequences.

Why should we care?

(1) Lots of applications:
- Analysis: harmonic, functional, global.
- Geometry: euclidian, hyperbolic, algebraic, differential.
- Topology: general, algebraic, differential.
- Number theory: analytic, algebraic, automorphic.
- Group theory: Lie groups, representation theory.
- Combinatorics.
- Sciences.
- Engineering.
- ...

(2) Some concrete applications:
- $\displaystyle\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx=\pi.$
- $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$
- Fundamental theorem of algebra.
- Counting prime numbers (sketch):

Consider

$$\pi(x)=\text{Number of prime numbers up to } x,$$
$$\mathrm{li}(x)=\int_0^x\frac{dt}{\log(t)}.$$

Then [MT]

$$|\pi(x)-\mathrm{li}(x)|<xe^{-0.39\sqrt{\log(x)}},\quad x\geq 2$$

The Riemann Hypothesis is equivalent to

$$|\pi(x)-\mathrm{li}(x)|<\sqrt{x}\log(x),\quad x\geq 2.$$

We define

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \mathrm{Re}(s)>1.$$

Euler discovered the product formula

$$\begin{aligned}\zeta(s)&=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\dots\\&=\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{2^{2s}}+\frac{1}{2^{3s}}+\dots\right)\left(\frac{1}{1^s}+\frac{1}{3^s}+\frac{1}{3^{2s}}+\frac{1}{3^{3s}}+\dots\right)\left(\frac{1}{1^s}+\frac{1}{5^s}+\frac{1}{5^{2s}}+\frac{1}{5^{3s}}+\dots\right)\dots\\&=\prod_{p\text{ prime}}\frac{1}{1-\frac{1}{p^s}}\qquad(\mathrm{Re}(s)>1).\end{aligned}$$

(3) Can explain phenomenons in real analysis: Consider the following smooth real functions and their Taylor series at $x=0$:

(a) $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots,$
(b) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots, \quad |x| < 1,$
(c) $e^{-1/x^2} = 1 + 0 + 0 + 0 + \dots$
Why is the radius of convergence different in each example?

1.2. **Arithmetic of the complex plane.** We can identify $\mathbb{C}$ with $\mathbb{R}^2$ and think of it as the Euclidiean plan:

$$\mathbb{C} \longleftrightarrow \mathbb{R}^2, \quad z = x + iy \mapsto \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Addition: we add complex numbers as vectors.
$$(x + iy) + (x' + iy') := (x + x') + i(y + y').$$
- Multiplication: we multiply two complex numbers by multiplying their lengths and adding their angles. Let us denote $\cos(t) + i\sin(t)$ by e^{it}. Then
$$(re^{it})(r'e^{it'}) := (rr')(e^{i(t+t')}).$$
This is called *polar coordinates.*

Remark. Usually, multiplication is defined in a different way, using Cartesian coordinates:
$$(x + iy)(x' + iy') := xx' - yy' + i(xy' + x'y).$$
Is this the same? Yes! One way to show this is to prove the law of distribution:
$$(w + w')z = wz + w'z,$$
$$w(z + z') = wz + wz'.$$
Multiplication by z is a rotation by the angle of z and a dilatation by its length. It preserves the triangle given by the vectors w, w' and $w + w'$. So the above equations are true. Then we have
$$(x + iy)(x' + iy') = xx' + (iy)(iy') + x(iy') + (iy)x' = xx' - yy' + i(xy' + x'y).$$

Remark. Another way to think of complex multiplication is the map application on smartphones (e.g. Google maps). Imagine you zoom on the map with two fingers in the following way: you start with your left finger on 0 and your right finger on 1. Then you move your right finger to z. The number zz' is given by the position of z' after this movement.

Theorem 1.1 (Just for fun, not legit)**.** *Let $a \le b \le c$ be the side lengths of a right triangle. Then*
$$a^2 + b^2 = c^2.$$

Proof. Without loss of generality, the triangle has vertices $0, a, a + ib$. Then
$$a^2 + b^2 = (a + ib)(a - ib) = (ce^{it})(ce^{-it}) = c^2.$$
□

Theorem 1.2. *Take a triangle and on each side, draw a regular triangle pointing outwards. Then the centers of the regular triangles form a regular triangle.*

Proof. We write a, b, c for the sides of the initial triangle and a', b', c' for the final triangle, with a' on the side opposed to a. Let $w := e^{2\pi i/3}$. Then by properties of the regular triangles on the sides of the initial triangle, we have
$$w(c - a') = b - a',$$
$$w(a - b') = c - b',$$
$$w(b - c') = a - c'.$$

Our goal is to prove $w(b'-a') = c'-b'$. From the equations above, we have

$$a' = \frac{b-wc}{1-w}, \quad b' = \frac{c-wa}{1-w}, \quad c' = \frac{a-wb}{1-w}.$$

So we want to show that

$$w\frac{(c-wa)-(b-wc)}{1-w} = \frac{(a-wb)-(c-wa)}{1-w}.$$

Equivalently

$$\begin{aligned} w((1+w)c - wa - b) &= (1+w)a - wb - c, \\ (w+w^2)c - w^2 a &= (1+w)a - c, \\ (1+w+w^2)c &= (1+w+w^2)a, \\ 1+w+w^2 &= 0. \end{aligned}$$

The last line is clear by the definition of w. □

Theorem 1.3. *Let $A_0, A_1, \dots, A_{n-1}$ be the vertices of a regular n-gon inscribed in the unit circle. Then*

$$A_0A_1 \cdot A_0A_2 \cdot \dots \cdot A_0A_{n-1} = n,$$

where we multiply the lengths of the vectors in the above equation.

Proof. Without loss of generality, the vertices are the n-th roots of unity: $A_k := e^{2\pi i k/n}$, $k = 0, \dots, n-1$. In other words, if $w := e^{2\pi i/n}$, we have $A_k = w^k$ for all k. We want to show

$$\prod_{k=1}^{n-1} |1 - w^k| = n.$$

Idea: replace 1 by a variable z and consider more generally the polynomial $\prod_{k=1}^{n-1}(z - w^k) \in \mathbb{C}[z]$. It is a monic polynomial with simple roots $w, w^2, \dots, w^{n-1}$. Note that $z^n - 1$ is a polynomial whose roots are all the n-th roots of unity $(1, w, w^2, \dots, w^{n-1})$. In other words

$$\begin{aligned} z^n - 1 &= \prod_{k=0}^{n-1}(z - w^k), \\ \frac{z^n-1}{z-1} &= \prod_{k=1}^{n-1}(z - w^k), \quad z \neq 1, \\ 1 + z + z^2 + \dots + z^{n-1} &= \prod_{k=1}^{n-1}(z - w^k). \end{aligned}$$

Now, replace z by 1 in the last equation:

$$n = \prod_{k=1}^{n-1}(1 - w^k).$$

□

Theorem 1.4. *Let $A_0, A_1, \dots, A_{n-1}$ be any n-gon inscribed in the unit circle. Let P run through the unit circle. Then*

$$\max_{|P|=1}(PA_0 \cdot PA_1 \cdot \dots \cdot PA_{n-1}) \geq 2.$$

Moreover, we have equality if and only if the n-gon is regular.

Proof. We can reformulate this problem as follow. Let

$$p(z) = \sum_{k=0}^{n} c_k z^k$$

be a monic polynomial of degree n ($c_n = 1$) whose n roots all lie on the unit circle. Then

$$\max_{|z|=1} |p(z)| \geq 2$$

with equality if and only if $p(z) = z^n + c_0$. This was first proved by Visser in 1945 [VV].

Let $w := e^{2\pi i/n}$ and fix z_0 on the unit circle. Consider

$$\frac{1}{n} \sum_{m=0}^{n-1} |p(z_0 w^m)|^2 .$$

First, we have

$$p(z_0 w^m) = \sum_{k=0}^{n} c_k z_0^k w^{mk} = \sum_{k=0}^{n-1} b_k w^{mk}$$

with

$$b_k = \begin{cases} c_k z_0^k & \text{if } 1 \leq k \leq n-1, \\ z_0^n + c_0 & \text{if } k = 0. \end{cases}$$

So

$$\begin{aligned} \frac{1}{n} \sum_{m=0}^{n-1} |p(z_0 w^m)|^2 &= \frac{1}{n} \sum_{m=0}^{n-1} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} b_k \bar{b}_l w^{m(k-l)} \\ &= \sum_{k=0}^{n-1} |b_k|^2 \\ &\geq |b_0|^2 \\ &= |z_0^n + c_0|^2 . \end{aligned}$$

We used that

$$\frac{1}{n} \sum_{m=0}^{n-1} w^{mr} = \begin{cases} 1 & \text{if } n \mid r, \\ 0 & \text{else.} \end{cases}$$

Therefore

$$\max_{|z|=1} |p(z)| \geq \max_{|z|=1} \left| \frac{1}{n} \sum_{m=0}^{n-1} |p(z_0 w^m)|^2 \right| \geq \max_{|z|=1} |z^n + c_0| = 2.$$

□

1.3. **Topology of the complex plane.** See first Sections 1.2 and 1.3 in the book.

Theorem 1.5 (Cantor's intersection theorem)**.** *Let $F_1 \supseteq F_2 \supseteq F_3 \subseteq \dots$ be a decreasing sequence of non-empty closed sets. If* $\operatorname{diam} F_n \to 0$*, then*

$$\bigcap_{n=1}^{\infty} F_n$$

is a singleton $\{z\}$*.*

Proof. We pick a point $z_n \in F_n$ for each n. Since $\operatorname{diam} F_n \to 0$, we have $|z_n - z_m| \leq \operatorname{diam} F_{\min(m,n)} \to 0$. It follows that the sequence (z_n) is Cauchy and converges to a point z.

Claim: $z \in \bigcap_{n=1}^{\infty} F_n$. It suffices to check that $z \in F_n$ for all n. Since the tail sequence $z_n, z_{n+1}, z_{n+2}, \dots$ is contained in F_n and F_n is closed, this is clear.

Finally, since $\operatorname{diam}(\bigcap_{n=1}^{\infty} F_n) \leq \operatorname{diam} F_n$ for all n, the diameter of the intersection is 0. So there can be at most one point in it. □

Theorem 1.6. *Let K be compact and $f : K \to \mathbb{C}$ be continuous. Then*

(1) *The set $f(K)$ is compact.*
(2) *The function f is uniformly continuous.*

Proof.

(1) Let $(z_k) \subseteq f(K)$ be a sequence in the image of f. There are $w_k \in K$ such that $f(w_k) = z_k$ for all k. Since K is compact, there exists a subsequence w_{k_j} that converges to $w \in K$. By definition of continuity, if $w_{k_j} \to w$, then $z_{k_j} = f(w_{k_j}) \to f(w) \in f(K)$. So we found a converging subsequence in $f(K)$ of (z_k).
(2) By contradiction, suppose that there exists $\epsilon > 0$ such that
$$\forall \delta > 0\ \exists w, z \in K : |w - z| < \delta \text{ and } |f(w) - f(z)| \geq \epsilon.$$
For $\delta_n = \frac{1}{n}$, pick two such w_n, z_n satisfying the above equations. Since $(w_n) \subseteq K$, there exists a subsequence $w_{n_j} \to w \in K$. Since $\left|w_{n_j} - z_{n_j}\right| < \frac{1}{n}$, $z_{n_j} \to w$ as well. Since f is continuous, we have $f(w_{n_j}) \to f(w) \leftarrow f(z_{n_j})$. It follows that $\left|f(w_{n_j}) - f(z_{n_j})\right| \to 0$, contradicting the above hypothesis $\left|f(w_{n_j}) - f(z_{n_j})\right| \geq \epsilon$ for all j.

□

Theorem 1.7. *Let $K \subseteq \mathbb{C}$ be a compact set and $\mathcal{F}$ an open cover of K. Then*
$$\exists r > 0\ \forall z \in K\ \exists U \in \mathcal{F} : D(z, r) \subseteq U.$$
In particular, if $\mathcal{F} = \{U\}$ for an open set U, then
$$\exists r > 0\ \forall z \in K : D(z, r) \subseteq U.$$

Proof. By contradiction, for each $r = \frac{1}{n}$, there exists a $z_n \in K$ such that $D(z_n, \frac{1}{n}) \not\subseteq U$ for any $U \in \mathcal{F}$. Since K is compact, there exists a convergent subsequence $z_{n_j} \to z \in K$. Pick $U \in \mathcal{F}$ and $r > 0$ such that $D(z, r) \subseteq U$. Finally, pick j such that $\left|z_{n_j} - z\right| < \frac{r}{2}$ and $\frac{1}{n_j} < \frac{r}{2}$. Then $D(z_{n_j}, \frac{1}{n_j}) \subseteq D(z, r)$. Contradiction. □

Definition 1.8. Let $M \subseteq \mathbb{C}$ be an open set. Then

- The set M is *connected* if it cannot be written as a disjoint union of two non-empty open sets.
- The set M is *path-connected* if every two points in M can be joined by a *continuous curve* lying in M. That is, for all $w, z \in M$, there exists a continuous function $u : [a, b] \to M$, $a < b$, such that $u(a) = w$ and $u(b) = z$.

Theorem 1.9. *Let $M \subseteq \mathbb{C}$ be an open set. The following are equivalent:*

(1) *The set M is connected.*
(2) *The set M is path-connected via broken lines consisting only of horizontal and vertical line segments.*
(3) *The set M is path-connected.*

Proof. (1) $\Rightarrow$ (2) : assume that M is connected. Let $z \in M$. We show that every point in M can be connected to z along a horizontal-vertical broken line that lies in M. First note that this is clear for $M = D(z, r)$: given $w = a + ib$ in $D(z, r)$, consider the broken going from z to a and from a to w.

Define M_1 to be the set of all points w connected to z along a horizontal-vertical broken line and $M_2 = M \backslash M_1$. Clearly $z \in M_1 \neq \emptyset$. Since $M = M_1 \cup M_2$, it suffices to show that M_1 and M_2 are open. Let $w \in M_1$ and u a horizontal-vertical broken line going from z to w. There exists $r > 0$ such that $D(w, r) \subseteq M$. Let $w' \in D(w, r)$. We get a path from z to w by joining u and the path given above. So $D(w, r) \subseteq M_1$.

Similarly, if $w \in M_2$, consider $r > 0$ such that $D(w, r) \subseteq M$. If there exists $w' \in D(w, r)$ such that $w' \in M_1$, then we can join the path from z to w' to the path from w to w' to show that $w \in M_1$. Contradiction. So $D(w, r) \subseteq M_2$. Since $M = M_1 \cup M_2$, both sets are open and $M_1 \neq \emptyset$, the only possibility is $M = M_1$.

(2) $\Rightarrow$ (3) : obvious since broken lines are continuous curves.

(3) $\Rightarrow$ (1) : by contradiction, assume that we can write $M = U_1 \cup U_2$ with U_1, U_2 open and disjoint. Pick $z_1 \in U_1$, $z_2 \in U_2$. By assumption, M is path-connected, so there exists a continuous curve $u : [a, b] \to M$ with $u(a) = z_1$ and $u(b) = z_2$. Let

$$t^* := \sup\{t \in [a, b] : u(t) \in U_1\}.$$

Then either $u(t^*) \in U_1$ or $u(t^*) \in U_2$. Both options lead to a contradiction: suppose that $u(t^*) \in U_1$. In particular, $t^* \neq b$. Since U_1 is open, there exists $\epsilon > 0$ such that $D(u(t^*), \epsilon) \subseteq U_1$. Since u is continuous, there exists $\delta > 0$ such that $t^* + \delta < b$ and

$$|t - t^*| < \delta \Rightarrow |u(t) - u(t^*)| < \epsilon.$$

In particular, $u(t + \delta/2) \in D(u(t^*), \epsilon) \subseteq U_1$. This contradicts the definition of t^*. A similar argument gives a contradiction for $u(t^*) \in U_2$. □

2. Complex functions

Book: pages 8 to 24.

2.1. Big O notation.

Let $f, g : \mathbb{C} \to \mathbb{C}$ be complex functions and $z_0 \in \mathbb{C}$. We write

$$f = O(g) \quad \text{as } z \to z_0,$$

if there exists a constant $C > 0$ such that $|f(z)| \leq C\,|g(z)|$ for $|z - z_0|$ small enough. We write

$$f = o(g) \quad \text{as } z \to z_0,$$

if for any constant $\epsilon > 0$, we have $|f(z)| \leq \epsilon\,|g(z)|$ for $|z - z_0|$ small enough. It is also possible to define the notations for $z_0 = \infty$ by considering all z big enough.

If $g(z) \neq 0$ for $|z - z_0|$ small enough, then

$$f = O(g) \quad \Leftrightarrow \quad \limsup_{z \to z_0} \left|\frac{f(z)}{g(z)}\right| < \infty,$$

$$f = o(g) \quad \Leftrightarrow \quad \lim_{z \to z_0} \frac{f(z)}{g(z)} = 0.$$

Example 2.1.

(1) $f = O(f)$.
(2) $f = O(1)$ if f is a bounded function in a neighborhood of z_0. E.g. $10^{100} = O(1)$.
(3) $f = o(1)$ if $\lim_{z \to z_o} f(z) = 0$.
(4) $1, z, 1000z^2, z^2 + 2z, z^2 + \sin(z)$ (last one for z real) are all $O(z^2)$ as $z \to \infty$.
(5) If $f = O(g)$ and h is another function, then $fh = O(gh)$.
(6) It is common to write $f = g + O(h)$ to mean $f - g = O(h)$ or even with more complex operations. E.g. if $f(z) = (z + 1)^2$ and $g(z) = z^2$, then

$$f(z) = g(z) + O(2z + 1) = g(z) + O(z) = g(z) + o(g(z)) = g(z)(1 + o(1)).$$

(7) If $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ is a converging power series, then

$$f(z) = \sum_{n=0}^{N} a_n (z - a)^n + O((z - a)^{N+1})$$

as $z \to a$. In particular, $f(z) = a_0 + O(z - a)$ More on this later.

We also write $f \sim g$ if $f = g(1 + o(1))$ or equivalently $f - g = o(g)$.

2.2. **Cauchy-Riemann equations.** We compare f being complex differentiable and real differentiable.

Definition 2.2. Let $M \subseteq \mathbb{R}^2$ be open. A function $f : M \to \mathbb{R}^2$ is *differentiable* if for every $\mathbf{x} \in M$, we have

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + D_{\mathbf{x}}\mathbf{h} + o(\|h\|)$$

where $D_{\mathbf{x}} \in \mathrm{M}_2(\mathbb{R})$. The linear map $\mathbf{h} \mapsto D_{\mathbf{x}}\mathbf{h}$ is the *derivative* of f at $\mathbf{x}$ and the matrix $D_{\mathbf{x}}$ is the *Jacobian* of f. If we take $\mathbf{h} = h\begin{pmatrix}1\\0\end{pmatrix}$ or $h\begin{pmatrix}0\\1\end{pmatrix}$ with $h \to 0$, we see that each coordinate function of $f = \begin{pmatrix}f_1\\f_2\end{pmatrix}$ is differentiable with respect to each coordinate of $\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$. Moreover

$$D_{\mathbf{x}} = \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix}.$$

Now, we consider a holomorphic function $f : M \in \mathbb{C} \cong \mathbb{R}^2$. It is clearly real differentiable. Moreover, the Jacobian matrix corresponds to the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by multiplication by $f'(z)$. Write $f'(z) = a + ib$. Then the image of 1 and i under multiplication by $f'(z)$ is given by

$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix}.$$

So there is a relationship between the partial derivatives of f.

Theorem 2.3 (Cauchy-Riemann equations)**.** *Let $M \subseteq \mathbb{C}$ be open, $f : M \to \mathbb{C}$. Identify $\mathbb{C}$ with $\mathbb{R}^2$ and write $f(x+iy) = u(x,y) + iv(x,y)$, i.e.*

$$u = \operatorname{Re} f, \quad v = \operatorname{Im} f$$

viewed as real functions. Then f is complex differentiable if and only if f is real differentiable and

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned} \tag{2.1}$$

Proof. If f is complex differentiable, we saw above that the Jacobian at z is given by the multiplication by $f'(z)$ and it satisfies the Cauchy-Riemann equations. Conversely, suppose that f is real differentiable and Equations (2.1) hold. Let $z = x + iy$ and $h = h_1 + ih_2$. Coordinatewise we have

$$\begin{aligned}u(x+h_1, y+h_2) - u(x,y) &= h_1\frac{\partial u}{\partial x}(x,y) + h_2\frac{\partial u}{\partial y}(x,y) + o(\|\mathbf{h}\|),\\ v(x+h_1, y+h_2) - v(x,y) &= h_1\frac{\partial v}{\partial x}(x,y) + h_2\frac{\partial v}{\partial y}(x,y) + o(\|\mathbf{h}\|).\end{aligned}$$

Then we compute

$$\begin{aligned}f(z+h) - f(z) &= u(x+h_1, y+h_2) - u(x,y) + i(v(x+h_1,y+h_2) - v(x,y))\\ &= h_1\frac{\partial u}{\partial x}(x,y) - h_2\frac{\partial v}{\partial x}(x,y) + ih_1\frac{\partial v}{\partial x}(x,y) + ih_2\frac{\partial u}{\partial x}(x,y) + o(\|\mathbf{h}\|)\\ &= (h_1 + ih_2)\left(\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)\right) + o(\|\mathbf{h}\|)\\ &= hf'(z) + o(\|\mathbf{h}\|).\end{aligned}$$

where $f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$. Note also that $\|\mathbf{h}\| = |h|$. ∎

2.3. **Power series.**

Proposition 2.4. *If* $\sum_{n=0}^{\infty} |c_n|$ *converges, then* $\sum_{n=0}^{\infty} c_n$ *converges.*

Proof. Assume $\sum_{n=0}^{\infty} |c_n|$ converges. By the Cauchy criterion,

$$\forall \epsilon > 0 \ \exists N \ \forall L > M > N : \sum_{n=M}^{L} |c_n| < \epsilon.$$

By the triangular inequality, we have

$$\left| \sum_{n=M}^{L} c_n \right| \le \sum_{n=M}^{L} |c_n| < \epsilon.$$

So the Cauchy Criterion is true for the series $\sum_{n=0}^{\infty} c_n$. □

Definition 2.5. Let $a \in \mathbb{C}$ and $(a_n) \subseteq \mathbb{C}$. The associated *power series* to (a_n) with *center* a is

$$\sum_{n=0}^{\infty} a_n (z-a)^n.$$

Definition 2.6. Let $(a_n) \subseteq \mathbb{R}$ be a sequence of real numbers. We define

$$\limsup_{n\to\infty} a_n := \lim_{m\to\infty} \sup_{n\ge m} b_n,$$
$$\liminf_{n\to\infty} a_n := \lim_{m\to\infty} \inf_{n\ge m} b_n.$$

The two sequences inside the limits on the right-hand sides are decreasing, resp. increasing. Therefore, the limits are well define in $\mathbb{R} \cup \{\pm\infty\}$.

It turns out that a power series always defines a holomorphic function in the interior of its domain. The converse, that a holomorphic is a power series will be an important result that we will see later.

Theorem 2.7 (Cauchy-Hadamard formula)**.** *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ *be a power series centered at* a*. Let*

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}. \tag{2.2}$$

Then, for $z \in \mathbb{C}$.

1. *If* $|z-a| > R$*, then* $f(z)$ *diverges. In fact, there exists a subsequence of the terms in* $f(z)$ *that grows exponentially fast.*
2. *If* $|z-a| < R$*, then* $f(z)$ *converges. In fact, the terms in* $f(z)$ *are bounded from above by a convergent geometric series. Hence they decay exponentially fast and* $f(z)$ *converges absolutely.*

Theorem 2.8. *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ *be a power series centered at* a*. Let*

$$g(z) := \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

be the formal derivative of f*.*

1. *The radius of convergence of* $f(z)$ *and* $g(z)$ *are the same.*
2. *If the common radius of convergence* R *is positive, then* $f(z)$ *defines a holomorphic function on* $D(a, R)$ *with derivative* $f'(z) = g(z)$*.*

Proposition 2.9. *Let* $(b_n) \subseteq \mathbb{R}$ *be a sequence of real numbers and* $L \in \mathbb{R}$*.*

1. *If* $L > \limsup_{n\to\infty} b_n$*, then the set* $\{n : b_n > L\}$ *is finite.*
2. *If* $L < \limsup_{n\to\infty} b_n$*, then the set* $\{n : b_n > L\}$ *is infinite.*
3. *If* $L < \liminf_{n\to\infty} b_n$*, then the set* $\{n : b_n < L\}$ *is finite.*
4. *If* $L > \liminf_{n\to\infty} b_n$*, then the set* $\{n : b_n < L\}$ *is infinite.*

Proposition 2.10. *Let $(b_n) \subseteq \mathbb{R}$ be a sequence of real numbers. Let $\mathcal{L} \subseteq \mathbb{R} \cup \{\pm\infty\}$ be the set of possible limits of all subsequences of (b_n). Then $\max \mathcal{L} = \limsup_{n\to\infty} b_n$ and $\min \mathcal{L} = \liminf_{n\to\infty} b_n$. In particular, the limsup/liminf of the sequence is in $\mathcal{L}$.*

Proof of Theorem 2.7. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ and

$$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

as in theorem 2.7.

(1) Without loss of generality $R < \infty$. Assume $|z-a| > R$, i.e.

$$|z-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1.$$

This is valid since $R < \infty$ implies that the limsup is non-zero. Let L be such that $1 < L < |z-a| \limsup \sqrt[n]{|a_n|}$. Then

$$1 < L < \limsup_{n\to\infty} \sqrt[n]{|a_n(z-a)^n|}$$

and $L < \sqrt[n]{|a_n(z-a)^n|}$ for infinitely many n by Proposition 2.9. Then $|a_n(z-a)^n| > L^n$ and the right-hand side grows exponentially fast.

(2) Without loss of generality $R > 0$. Assume $|z-a| < R$. Then

$$|z-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1.$$

As before, let L be such that $|z-a| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < L < 1$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(z-a)^n|} < L < 1.$$

By Proposition 2.9, $\sqrt[n]{|a_n(z-a)^n|} > L$ only for finitely many n. That is, there exists N such that

$$|a_n(z-a)^n| \leq L^n$$

for all $n \geq N$. Then

$$f(z) = \sum_{n=0}^{N-1} a_n(z-a)^n + \sum_{n=N}^{\infty} a_n(z-a)^n.$$

The first term is a finite sum and the second term is a series majorized by $\sum_{n=0}^{\infty} L^n < \infty$. By Proposition 2.4, $f(z)$ converges absolutely.

□

Proof of Theorem 2.8.

(1) We want to show that f and g have the same radius of convergence. First, note that $g(z)$ converges if and only if $(z-a)g(z)$ converges. This is clear if $z = a$, and if $z \neq a$, $\frac{1}{z-a}$ exists and the equivalence is clear. It remains to show that $f(z)$ and $(z-a)g(z)$ have the same radius of convergence, that is

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|na_n|}.$$

This follows from $\sqrt[n]{n} \to 1$ as $n \to \infty$.

(2) Assume that the common radius of convergence R of $f(z)$ and $g(z)$ is positive. Let $|z-a| < R$. We want to show that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = g(z).$$

We fix z and a positive number r such that

$$|z-a| < r < R.$$

Without loss of generality, h satisfies

$$0 < |h| < r - |z-a|.$$

Then we have $|z+h-a| \leq |z-a| + |h| < r$. Let also $\epsilon > 0$ be fixed. We separate the series of f and g with respect to a certain N to be defined later. That is, we write

$$f(z) = f_1(z) + f_2(z) = \sum_{n=0}^{N-1} a_n(z-a)^n + \sum_{n=N}^{\infty} a_n(z-a)^n,$$
$$g(z) = g_1(z) + g_2(z) = \sum_{n=1}^{N-1} na_n(z-a)^{n-1} + \sum_{n=N}^{\infty} na_n(z-a)^{n-1}.$$

The first part of the series is called the partial sum and the second part is called the tail. We write

$$\begin{aligned} \left|\frac{f(z+h)-f(z)}{h} - g(z)\right| &= \left|\frac{f_1(z+h)+f_2(z+h)-f_1(z)-f_2(z)}{h} - g_1(z) - g_2(z)\right| \\ &\leq \left|\frac{f_1(z+h)-f_1(z)}{h} - g_1(z)\right| + \left|\frac{f_2(z+h)-f_2(z)}{h}\right| + |g_2(z)|. \end{aligned} \tag{2.3}$$

We want to show that each term in the last line is smaller than $\frac{\epsilon}{3}$ for h small enough. The first term only consists of finite sums. More precisely, f_1 is a polynomial in z, and as such its derivative is

$$f_1'(z) = \sum_{n=1}^{N-1} na_n(z-a)^{n-1} = g_1(z).$$

By definition of the derivative, there exists $0 < \delta < r - |z-a|$ such that

$$\left|\frac{f_1(z+h)-f_1(z)}{h} - g_1(z)\right| < \frac{\epsilon}{3}$$

for all $0 < |h| < \delta$. The second term in Equation 2.3 is

$$\left|\frac{f_2(z+h)-f_2(z)}{h}\right| = \left|\sum_{n=N}^{\infty} a_n \frac{(z+h-a)^n-(z-a)^n}{h}\right| \leq \sum_{n=N}^{\infty} |a_n| \left|\frac{(z+h-a)^n-(z-a)^n}{h}\right|.$$

Note that $(z+h-a)-(z-a) = h$. So the fraction corresponds to the sum

$$\frac{(z+h-a)^n-(z-a)^n}{h} = \sum_{j=0}^{n-1} (z+h-a)^j (z-a)^{n-1-j}.$$

Each term in the sum is bounded by $\max\{|z+h-a|, |z-a|\}^{n-1} \leq r^{n-1}$. Then

$$\left|\frac{f_2(z+h)-f_2(z)}{h}\right| \leq \sum_{n>N} |a_n| \left|\frac{(z+h-a)^n-(z-a)^n}{h}\right| \leq \sum_{n=N}^{\infty} n\,|a_n|\, r^n.$$

This looks a lot to $g_2(z)$. More precisely, consider

$$g(r+a) = \sum_{n=1}^{\infty} na_n r^{n-1}.$$

Since $r+a$ lies in $D(a,R)$, $g(r+a)$ converges absolutely (by Theorem 2.7). So for N big enough, we have

$$\sum_{n=N}^{\infty} n\,|a_n|\, r^{n-1} < \frac{\epsilon}{3}.$$

Finally, the last term in Equation 2.3 is

$$|g_2(z)| \leq \sum_{n=N}^{\infty} n\,|a_n|\,|z-a|^{n-1} \leq \sum_{n=N}^{\infty} n\,|a_n|\, r^{n-1} < \frac{\epsilon}{3}.$$

In summary, given $z \in D(a,R)$ and $\epsilon > 0$, we found a $\delta > 0$ such that for all $0 < |h| < \delta$, we have

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \leq \left|\frac{f_1(z+h)-f_1(z)}{h} - g_1(z)\right| + \left|\frac{f_2(z+h)-f_2(z)}{h}\right| + |g_2(z)| \tag{2.4}$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leq \epsilon \tag{2.5}$$

That is the derivative of f is g.

□

Corollary 2.11. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ be a series with a positive radius of convergence $R > 0$. Then $f(z)$ is infinitely many times complex differentiable in $D(a,R)$ and $a_n = \frac{f^{(n)}(a)}{n!}$. In short, f is equal to its Taylor series at a.*

Proof. By induction on k, Theorem 2.8 says that $f^{(k)}$ exists and is equal to

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}.$$

Hence $f^{(k)}(a) = k(k-1)\cdots(k-k+1)a_n = k!a_n$. □

Definition 2.12.

$$\begin{aligned}
e^z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots, \\
\sin(z) &:= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{6} + \frac{z^5}{120} + \dots, \\
\cos(z) &:= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots, \\
\sinh(z) &:= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{6} + \frac{z^5}{120} + \dots, \\
\cosh(z) &:= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2} + \frac{z^4}{24} + \dots
\end{aligned}$$

Proposition 2.13.

(1) *The functions e^z, $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$ are all holomorphic functions on all of $\mathbb{C}$.*

(2)

$$\begin{aligned}
(e^z)' &= e^z, & & \\
\sin(z)' &= \cos(z), & \cos(z)' &= -\sin(z), \\
\sinh(z)' &= \cosh(z), & \cosh(z) &= \sinh(z).
\end{aligned}$$

(3)

$$\begin{aligned}
e^{iz} &= \cos(z) + i\sin(z), & & \\
\sin(iz) &= i\sinh(z), & \cos(iz) &= \cosh(z), \\
\sin(z) &= \frac{e^{iz}-e^{-iz}}{2i}, & \cos(z) &= \frac{e^{iz}+e^{-iz}}{2}, \\
\sinh(z) &= \frac{e^z - e^{-z}}{2}, & \cosh(z) &= \frac{e^z+e^{-z}}{2}.
\end{aligned}$$

Lemma 2.14 (Stirling's formula, soft version)**.** *We have*

$$n! = \left(\frac{n}{e}\right)^n e^{O(\log(n))} = \left(\frac{n}{e}\right)^n n^{O(1)}.$$

Proof. Consider $\log(n!) = \log(1) + \log(2) + \cdots + \log(n)$. We can compare it to an integral

$$\sum_{k=1}^{n} \log(k) \leq \sum_{k=1}^{n} \int_{k}^{k+1} \log(t)dt = \int_{1}^{n+1} \log(t)dt = (n+1)\log(n+1) - (n+1) - 1,$$

$$\sum_{k=1}^{n} \log(k) \geq \log(1) + \sum_{k=2}^{n} \int_{k-1}^{k} \log(t)dt = \int_{1}^{n} \log(t)dt = n\log(n) - n - 1.$$

Note that $\log(n+1) = \log(n) + \log(1+\frac{1}{n}) = \log(n) + O(\frac{1}{n})$. We get

$$\log(n!) = n\log(n) - n + O(\log(n)).$$

Finally, taking the exponential, we get

$$n! = \left(\frac{n}{e}\right)^n e^{O(\log(n))}.$$

□

Proof.

(1) It is enough, by Theorems 2.7 and 2.8, to show that the series have an infinite radius of convergence. For e^z, we want to show that

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}}} = \infty,$$

that is $\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0$. By the lemma above, we have

$$\sqrt[n]{n!} = \frac{n}{e} e^{O\left(\frac{\log(n)}{n}\right)} \to \infty.$$

The proof for the other series is similar.

(2) We proof the formula for e^z. The proof for the other series is similar.

$$(e^z)' = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = e^z.$$

(3) The identities are direct consequences of the definitions. For the first one, we have

$$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = \sum_{n=2k \text{ even}} \frac{(-1)^k z^{2k}}{(2k)!} + \sum_{n=2k+1 \text{ odd}} \frac{(-1)^k i z^{2k+1}}{(2k+1)!} = \cos(z) + i\sin(z).$$

We also get that

$$e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z).$$

Then

$$e^{iz} - e^{-iz} = 2i\sin(z), \qquad e^{iz} + e^{-iz} = 2\cos(z).$$

The formulas for $\cosh(z)$ and $\sinh(z)$ are straightforward from the definitions and the other results.

□

Proposition 2.15. *Let $M \subseteq \mathbb{C}$ a connected open set. Let $f : M \to \mathbb{C}$. Then f is constant if and only if $f' = 0$.*

Proof. If f is constant, then $f' = 0$ by definition. Suppose that $f' = 0$. Write $f = u + iv$ with $u, v : M \to \mathbb{R}$ in the usual way. Then the Jacobian matrix of f vanishes:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = 0.$$

By the same result for real functions, u and v are constant on horizontal and vertical lines contained in M. By Theorem 1.9, any $z_1, z_2 \in M$ can be connected by a broken line consisting of horizontal and vertical paths. So $u(z_1) = u(z_2)$ and the same is true for v. In conclusion, f is a constant function. ☐

Proposition 2.16. *Let $a, b \in \mathbb{C}$. Then*
$$e^{a+b} = e^a e^b.$$

Proof. Equivalently, $e^z e^{w-z} = e^w$. Fix w and consider $f(z) = e^z e^{w-z}$. We want to show that f is constant. By Proposition 2.15, we need to show that $f' = 0$. We have
$$f'(z) = (e^z e^{w-z})' = e^z e^{w-z} + e^z(-e^{w-z}) = 0.$$
So $f(z) = c$ is constant. Moreover, $f(0) = e^0 e^{w-0} = e^w$ so $c = e^w$. ☐

Corollary 2.17.

(1) $e^{-z} = \frac{1}{e^z}$.
(2) $e^z \neq 0$ *for all* $z \in \mathbb{C}$.
(3) $\left|e^{it}\right| = 1$ *for* $t \in \mathbb{R}$.

2.4. **Integration along curves.** See the book, Section 3 in Chapter 1.

3. Cauchy's theorem

Book: pages 33 to 45. Skipping section 2.

3.1. Cauchy's theorem for convex regions.

Definition 3.1. A set $M \subseteq \mathbb{C}$ is convex if for all $z_1, z_2 \in M$, the segment $[z_1, z_2]$ is entirely in M.

Theorem 3.2. *Let $M \subseteq \mathbb{C}$ be a* convex *region set and $f : M \to \mathbb{C}$ be a continuous function. The following are equivalent:*

(1) *f has a primitive.*
(2) *$\int_\gamma f = 0$ for every closed curve $\gamma \subseteq M$.*
(3) *$\int_\gamma f = 0$ for every triangular contour $\gamma \subseteq M$.*

Proof of Theorem 3.2. 1) ⇒ 2): if f has a primitive, then the integral of f over a closed curve vanishes by the fundamental theorem of calculus.

2) ⇒ 3): trivial.

3) ⇒ 1): assume that $\int_\gamma f = 0$ for all triangular contour $\gamma \subseteq M$. Let $z_0 \in M$ be a fixed point. Since M is convex, we can define:
$$F(z) := \int_{[z_0,z]} f.$$
Claim: F is a primitive of f.

Let $z \in \mathbb{C}$ and $r > 0$ be fixed such that $D(z, r) \subseteq M$. Let $h \in \mathbb{C}$ with $0 < |h| < r$. Then by hypothesis
$$\int_{[z_0,z]} f + \int_{[z,h]} f + \int_{[h,z_0]} f = 0,$$
i.e.
$$F(z+h) = F(z) + \int_{[z,z+h]} f.$$
By continuity of f on $D(z, r)$
$$f(z+h) - f(z) = o(1)$$
as $h \to 0$. Then
$$\int_{[z,z+h]} f(w)dw = \int_{[z,z+h]} f(z)dw + \int_{[z,z+h]} (f(w) - f(z))dz = f(z)h + o(h).$$

We used that

$$\left| \int_{[z,z+h]} o(1) dw \right| \leq h \sup_{[z,z+h]} o(1) = o(h).$$

In conclusion

$$F(z+h) = F(z) + f(z)h + o(h).$$

So $f(z)$ is the derivative of $F(z)$. □

Theorem 3.3 (Goursat's lemma)**.** *Let $m \subseteq \mathbb{C}$ be a convex region and $f : M \to \mathbb{C}$ be holomorphic. Then for every triangular contour γ, the integral $\int_\gamma f$ vanishes.*

Proof. See the book. □

Corollary 3.4 (Cauchy)**.** *Let $M \subseteq \mathbb{C}$ be a convex region and $f : M \to \mathbb{C}$ be holomorphic. Then*

1. *The function f has a primitive.*
2. *The integral $\int_\gamma f = 0$ for every closed curve $\gamma \subseteq M$.*

Remark. Later we will see as a consequence of Cauchy's formula that only holomorphic functions can have a primitive.

3.2. **Cauchy's theorem for general regions.** See also the book, Sections 5 and 6 in Chapter 3.

Definition 3.5. A *free homotopy* of two closed curves γ_0, γ_1 in a region M is a continuous map $\gamma(\tau, t) : [0,1]^2 \to M$ such that

1. $\gamma(0,t) = \gamma_0(t)$.
2. $\gamma(1,t) = \gamma_1(t)$.
3. $\gamma(\tau, 0) = \gamma(\tau, 1)$ for all $\tau \in [0,1]$.

The curves γ_0 and γ_1 are said *freely homotopic* to each other. A *free homotopy class* is the set of all closed curves homotopic to a given curve γ_0.

Theorem 3.6. *Let $M \subseteq \mathbb{C}$ be an open connected set, $f : M \to \mathbb{C}$ holomorphic and $\gamma_0, \gamma_1 \subseteq M$ two closed curves that are freely homotopic to each other within M. Then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Remark. Cauchy's theorem for convex regions is a special case of this. Because if M is convex, γ_0 and γ_1 are always homotopic to each other via

$$\gamma(\tau, t) := \tau\gamma_0(t) + (1-\tau)\gamma_1(t).$$

By definition of convexity, $\gamma(\tau, t) \in M$ for all τ, t. In that case, the integral is always 0 actually.

Remark. In the book, Theorem 5.1 in Chapter 3 is about homotopic non-closed curves with the same endpoints. It is a consequence of our theorem, using that γ_1 join $\bar{\gamma}_0$ is a closed curve homotopic to γ_0 join $\bar{\gamma}_0$.

$$\int_{\gamma_1} f - \int_{\gamma_0} f = \int_{\gamma_1 \text{ join } \bar{\gamma}_0} f = \int_{\gamma_0 \text{ join } \bar{\gamma}_0} f = 0.$$

Proof. The goal is to write

$$\int_{\gamma_0} f - \int_{\gamma_1} f$$

as a sum of many integrals $\int_C$ with each C being a closed curve in an open disk lying in M. Applying Cauchy's theorem for convex regions, we conclude. To do that, we split $[0,1]^2$ into many small enough squares. By continuity, the images of all the squares can be made small enough to sit in a disk.

More precisely, let N to be fixed later. By hypothesis, there exists a free homotopy $\gamma : [0,1]^2 \to M$ between γ_0 and γ_1. We split $[0,1]^2$ into N times N squares and write

$$z_{i,j} = \gamma\left(\frac{i}{N}, \frac{j}{N}\right), \quad 1 \leq i,j \leq N-1.$$

For $1 \leq i \leq N-2$ and $0 \leq j \leq N-1$, we define

$$C_{i,j} = [z_{i,j}, z_{i,j+1}] \text{ join } [z_{i,j+1}, z_{i+1,j+1}] \text{ join } [z_{i+1}, z_{j+1}, z_{i+1,j}] \text{ join } [z_{i+1,j}, z_{i,j}].$$

For $i = 0$ and $i = N-1$, we modify the definition to follow the respective curve on the edge of the domain. That is

$$C_{0,j} = \gamma_0|_{[\frac{j}{N}, \frac{j+1}{N}]} \text{ join } [z_{0,j+1}, z_{1,j+1}] \text{ join } [z_1, z_{j+1}, z_{1,j}] \text{ join } [z_{1,j}, z_{0,j}]$$

$$C_{N-1,j} = [z_{N-1,j}, z_{N-1,j+1}] \text{ join } [z_{N-1,j+1}, z_{N,j+1}] \text{ join } \overline{\gamma_1|_{[\frac{j}{N}, \frac{j+1}{N}]}} \text{ join } [z_{N,j}, z_{N-1,j}].$$

In $C_{N-1,j}$, we had to reverse the curve γ_1 restricted to the relevant part, since we are following it backwards. It is clear from the setup that

$$\sum_{0 \leq i,j \leq N-1} \int_{C_{i,j}} f = \int_{\gamma_0} f + \int_{\bar{\gamma}_1} f = \int_{\gamma_0} f - \int_{\gamma_1} f.$$

Claim: for large enough N, each $C_{i,j}$ can be covered by an open disk in M.
Proof of claim: since $[0,1]^2$ is a compact set, so is $K := \gamma([0,1]^2) \subseteq K$ by Theorem 1.6. Moreover, there exists $r > 0$ such that $D(z,r) \subseteq M$ for all $z \in K$ by Theorem 1.7. Since γ is uniformly continuous on K, there exists δ such that

$$\forall z_1, z_2 \in [0,1]^2 : |z_1 - z_2| < \delta \Rightarrow |\gamma(z_1) - \gamma(z_2)| < r.$$

Consider $N > \frac{\sqrt{2}}{\delta}$. Then $\delta > \frac{\sqrt{2}}{N}$, so all squares in the $n \times n$ grid of $[0,1]^2$ have diameter less than δ. Then $C_{i,j} \subseteq D(z_{i,j}, r) \subseteq M$ for all $0 \leq i,j \leq n-1$. This proves the claim.

Now, Cauchy's theorem for convex regions (Corollary 3.4) applies to the curve $C_{i,j}$ sitting in a disk. So

$$\int_{C_{i,j}} f = 0$$

for all $0 \leq i,j \leq N-1$ and the theorem is proven. □

Definition 3.7. A region $M \subseteq \mathbb{C}$ is *simply connected* if any two closed curves are freely homotopic to each other in M. Equivalently, every closed curve is freely homotopic to a single point in M. We say that the curve can be *contracted*.

As a direct consequence.

Theorem 3.8 (Cauchy's theorem for simply connected regions)**.** *Let $M \subseteq \mathbb{C}$ be a simply connected region and $f : M \to \mathbb{C}$ be holomorphic. Then*

(1) *f has a primitive.*
(2) *$\int_\gamma f = 0$ for every closed curve $\gamma \subseteq M$.*

Example 3.9.

(1) A convex region is simply connected.
(2) A star region is simply connected.
(3) If M_1 and M_2 are homeomorphic, then M_1 is simply connected if and only if M_2 is simply connected. In particular, homeomorphic images of $D(0,1)$ are simply connected.
(4) Solid open angles are simply connected. In particular, the slit plane $\mathbb{C} \backslash \{(-\infty, 0]\}$ is simply connected.
(5) The punctered plane $\mathbb{C} \backslash \{0\}$ is NOT simply connected.

Remark.

(1) What is a simply connected region? How to describe it except from the definition? How to characterize it?
(2) Is there an even more general Cauchy theorem?

Theorem 3.10. *Let $M \subseteq \mathbb{C}$ be a region. The following are equivalent:*

(1) *M is simply connected.*

(2) *Every holomorphic function on M has a primitive.*
(3) *Every non-vanishing holomorphic function f on M has a holomorphic logarithm. That is, there exists a holomorphic function $g : M \to \mathbb{C}$ such that $f = e^g$.*
(4) *Every non-vanishing holomorphic function f on M has a holomorphic square-root. That is, there exists a holomorphic function $g : M \to \mathbb{C}$ such that $f = g^2$.*
(5) *Either $M = \mathbb{C}$ or there exists a holomorphic bijection $M \to D(0,1)$.*
(6) *M is homeomorphic to $D(0,1)$.*

Remark. The Riemann mapping theorem states that (1) $\Leftrightarrow$ (5).

Proof. (1) $\Rightarrow$ (2): Done.

(2) $\Rightarrow$ (3): We want to find $g : M \to \mathbb{C}$ holomorphic such that $f = e^g$. Equivalently:

$$\begin{aligned}
&\exists g : f = e^g,\\
\Leftrightarrow &\exists g : fe^{-g} = 1,\\
\Leftrightarrow &\exists g : fe^{-g} = cst,\\
\Leftrightarrow &\exists g : (fe^{-g})' = 0,\\
\Leftrightarrow &\exists g : f'e^{-g} = fg'e^{-g},\\
\Leftrightarrow &\exists g : g' = \frac{f'}{f}.
\end{aligned}$$

By applying 2), it suffices to show that $\frac{f'}{f}$ is holomorphic, that is f' is holomorphic. We will prove this in the next chapter using Cauchy's formula.

(3) $\Rightarrow$ (4): Take $e^{g/2}$ where g is the holomorphic logarithm of f.

(4) $\Rightarrow$ (5) (sketch): The key step. Assume (4) and that $M \neq \mathbb{C}$. We want to find a holomorphic bijection $M \to D(0,1)$. Without loss of generality, $0 \notin M$ and $1 \in M$. First we find an injection $f : M \to D(0,1)$ with $f(1) = 0$. Consider the holomorphic square-root function $g(z) = \sqrt{z} : M \to \mathbb{C}$ given by (4). We have $g(1)^2 = 1$. If $g(1) = -1$, replace g by $-g$. So WLOG $g(1) = 1$.

By continuity, there exists a disk $D(1,\rho)$ such that $g(re^{it}) = r^{1/2}e^{it/2}$ in $D(1,\rho)$. On that disk, g is a bijection with continuous inverse. So the image of $D(1,\rho)$ is open and contains 1. Thus the image of g contains a disk $D(1,\rho')$ (with $\rho' < 1$). Moreover, the image of g is disjoint from $D(-1,\rho')$ because it cannot contain the two roots of a given complex number. Define

$$f(z) = \frac{\rho'}{2}\left(\frac{1}{\sqrt{z}+1} - \frac{1}{2}\right).$$

Then $|\sqrt{z}+1| \geq \rho'$ implies that $\frac{1}{\sqrt{z}+1} \leq \frac{1}{\rho'}$ and so $|f(z)| \leq \frac{\rho'}{2}(\frac{1}{\rho'} + \frac{1}{2}) = \frac{1}{2} + \frac{\rho'}{4} \leq \frac{3}{4}$. Thus f defines a function from M to $D(0,1)$ and $f(1) = 0$.

The following is a consequence of Cauchy's formula.

Theorem 3.11 (Montel)**.** *Let (f_n) be a sequence of holomorphic functions from M to $D(0,1)$. There exists a subsequence (f_{n_k}) that converges uniformly on compact subsets of M.*

Remark. If f is the pointwise limit of (f_{n_k}), then $f'_{n_k} \to f'$ uniformly on compact subsets of M. Moreover it is possible to show that if f_{n_k} is injective and f is not constant, then f is a holomorphic injection $M \to D(0,1)$.

Consider the family $\mathcal{F}$ of all functions that are holomorphic injections $f : M \to D(0,1)$ with $f(1) = 0$. We know that $\mathcal{F} \neq \emptyset$. Consider $m = \sup_{f\in\mathcal{F}} |f'(1)|$. There is a sequence (f_n) such that $|f_n(1)| \to m$. By Montel's theorem, there exists a subsequence that converges, and by the remarks after, the pointwise limit f_0 is also a function in $\mathcal{F}$ (note that $f_0'(1) \neq 0$ so f_0 cannot be constant).
Claim: the function f_0 is surjective.
Proof of claim: (sketch) by contradiction, suppose that there exists $w \in D(0,1)$ not in the image of f_0. The function

$$g_w : \frac{w - z}{1 - \bar{w}z}$$

sends w to 0 and 0 to w. Moreover, $g_w \circ g_w(z) = z$ so it is a bijection. We consider $h = g_{\sqrt{w}} \circ \sqrt{z} \circ g_w$ on $f(M)$. We have $h(0) = 0$. Note that the square root is well defined since $g_w(f(M))$ does not contains 0 and is simply connected. Its inverse is given by

$$h^{-1}(z) = g_w \circ z^2 \circ g_{\sqrt{w}} = \frac{w - \left(\frac{\sqrt{w}-z}{1-\overline{\sqrt{w}}z}\right)^2}{1 - \bar{w}\left(\frac{\sqrt{w}-z}{1-\overline{\sqrt{w}}z}\right)^2} = -\frac{z^2(|w|+1) - 2\sqrt{w}z}{|w|+1-2\overline{\sqrt{w}}z}.$$

Then

$$(h^{-1})'(0) = -\left.\frac{z(|w|+1) - 2\sqrt{w}}{|w|+1-2\overline{\sqrt{w}}z}\right|_{z=0} = \frac{2\sqrt{w}}{|w|+1}$$

Note that $\left|(h^{-1})'(0)\right| = \frac{2|\sqrt{w}|}{1+|w|} < 1$ since $|w| < 1$ and $2x < 1+x^2$ for all $x \neq 1$. Then $h'(0) = \frac{1}{(h^{-1})'(0)} > 1$ and

$$(h \circ f_0)'(1) = h'(0)f_0'(1)$$

has absolute value bigger than $f_0'(1)$, a contradiction with its definition. So f_0 is surjective.

(5) ⇒ (6): If $M = \mathbb{C}$, done. Otherwise, by (5) we have a holomorphic function $f : M \to D(0,1)$. We want to show that its inverse is continuous. This follows from the open mapping theorem that we will prove later and the following characterization of continuity.

Lemma 3.12. *Let M, N be two open sets. A function $f : M \to N$ is continuous if and only if for all open set $U \subseteq N$, $f^{-1}(U)$ is open.*

Actually, one can show that f' doesn't vanishes and the inverse of f is holomorphic.

(6) ⇒ (1): This is an exercise in topology. Use that $D(0,1)$ is simply connected. □

The logarithm function $\log(x) : \mathbb{R}_{>0} \to \mathbb{R}$ can be extended to complex numbers. Write $\log(re^{it}) = \log(r) + it$. This is well defined up to $2\pi i$.

Definition 3.13. The *principal value* of the logarithm $\log : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ is defined on $z \neq 0$ by taking the argument of z in $(-\pi, \pi]$.

Theorem 3.14. *Let $M \subseteq \mathbb{C}$ be a simply connected region with $0 \notin M$ and $1 \in M$. Then there is a branch of the logarithm $\log_M : M \to \mathbb{C}$ such that:*

1. *$\log_M$ is holomorphic on M.*
2. *$e^{\log_M(z)} = z$ for all $z \in M$.*
3. *$\log_M(r) = \log(r)$ for r a real number close to 1.*

Proof. By theorem 3.10, the function $f(z) = z$ on M has a holomorphic logarithm. That is there exists $g : M \to \mathbb{C}$ holomorphic such that $e^{g(z)} = z$. Moreover $e^{g(r)} = r$ for $r \in \mathbb{R}_{>0} \cap M$, so $g(r) = \log(r) + 2\pi i k_r$ for some $k_r \in \mathbb{Z}$ depending on r. Consider $\tilde{g}(z) = g(z) - 2\pi i k_1$. This satisfies the first two conditions of the theorem and $\tilde{g}(1) = 1$. Since M is open, there is $R > 0$ such that $D(1,R) \subseteq M$. By continuity of $\tilde{g}$, we have $\tilde{g}(r) = \log(r)$ for $r \in D(1,r) \cap \mathbb{R}$.

Alternatively, one can directly define $g(z) = \int_{\gamma_z} \frac{dz}{z}$ for γ_z any curve from 1 to z. □

Remark. Note that for $z = x+iy$ and the principal value of the logarithm, $\log(e^z) = \log(e^x e^{iy}) = x+i\tilde{y}$, where $\tilde{y} = y \pmod{2\pi}$ is such that $\tilde{y} \in (-\pi, \pi]$.

In general, $\log(wz) \neq \log(w) + \log(z)$. Consider $w = z = e^{2\pi i/3}$. For the principal branch of the logarithm, $\log(w) + \log(z) = \frac{2\pi i}{3} + \frac{2\pi i}{3} = \frac{4\pi i}{3}$. But $\log(wz) = \log(e^{4\pi i/3}) = \log(e^{-2\pi i/3}) = -\frac{2\pi i}{3}$.

Definition 3.15. Let $\alpha \in \mathbb{C}$. We define the *power function*

$$z^\alpha := e^{\alpha \log(z)}, \quad \forall z \in \mathbb{C}\backslash\{0\}.$$

4. Cauchy's formula

Book: pages 45 to 55.

Example 4.1 (Analytic continuation)**.** Let $f(z) = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$. This function is holomorphic on $D(0, 1)$. Let $g(z) = \frac{1}{1-z}$. It is a holomorphic function on $\mathbb{C}\backslash\{0\}$. We know that $f(z) = g(z)$ on $D(0, 1)$. Therefore, $g(z)$ is an analytic continuation of $f(z)$.

4.1. **Uniform convergence of sequences of holomorphic functions.**

Recall the content of Section 5.2 in Chapter 2.

Example 4.2. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The Riemann Zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We show that this function is well defined and holomorphic. By Theorem 5.2, it suffices to show that the sequence of partial sums

$$S_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}$$

converges uniformly to $\zeta(s)$ on every compact in the half-plane $\mathrm{Re}(s) > 1$. Let K be such a compact. Note that $\mathrm{Re}(s) : \mathbb{C} \to \mathbb{R}$ is a continuous function. The image $\mathrm{Re}(K)$ is compact, i.e. a closed interval $[a, b]$ with $a > 1$.

We prove that S_N converges uniformly to ζ on half-planes $\mathrm{Re}(s) \geq a$ for all $a > 1$. This implies the uniform convergence on every compact. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq a$. Then

$$|\zeta(s) - S_n(s)| = \left| \sum_{n=N+1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=N+1}^{\infty} \frac{1}{n^{\mathrm{Re}(s)}} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^a}.$$

We compare the function $\frac{1}{n^a}$ to an integral:

$$\frac{1}{n^a} \leq \int_{n-1}^{n} \frac{dt}{t^a}$$

since $\frac{1}{t^a}$ is a decreasing function. Then

$$|\zeta(s) - S_n(s)| \leq \sum_{n=N+1}^{\infty} \int_{n-1}^{n} \frac{dt}{t^a} = \int_{N}^{\infty} \frac{dt}{t^a} = \left. \frac{t^{-a+1}}{-a+1} \right|_N^{\infty} = \frac{N^{1-a}}{a-1}.$$

Note that the final term converges to 0 as $N \to \infty$ and does not depend on s. So the convergence is uniform. By Theorem 5.2, Chapter 2, in the book, $\zeta(s)$ is holomorphic.

4.2. **The factorial and the Gamma function.**

Definition 4.3. Let $z \in \mathbb{C}$ with $\mathrm{Re}(s) > -1$. We define

$$\Pi(z) := \int_0^{\infty} e^{-t} t^z dt.$$

This defines a holomorphic function. The Gamma function is defined by $\Gamma(z) := \Pi(z - 1)$.

Proposition 4.4. *We have* $\Pi(z + 1) = (z + 1)\Pi(z)$*. In particular,* $\Pi(n) = n!$ *for* n *a non-negative integer.*

Proof. We integrate by parts:

$$\begin{aligned} \Pi(z+1) &= \int_0^{\infty} e^{-t} t^{z+1} dt \\ &= -e^{-t} t^z |_0^{\infty} + (z+1) \int_0^{\infty} e^{-t} t^z dt \\ &= (z+1)\Pi(z). \end{aligned}$$

Moreover $\Pi(0) = \int_0^\infty e^{-t}dt = 1 = 0!$. By induction, $\Pi(n) = n!$. □

It is possible to extend analytically $\Pi(z)$ to $\mathbb{C}\backslash\{-1,-2,-3,\dots\}$. Note that

$$\Pi_1(z) = \frac{\Pi(z+1)}{z+1}$$

is a function defined and holomorphic for $\mathrm{Re}(z) > -2$ and $z \neq -1$. Moreover, $\Pi_1(z) = \Pi(z)$ for $\mathrm{Re}(z) > -1$. It is the analytic continuation of $\Pi(z)$. Inductively, one can define

$$\Pi_m(z) = \frac{\Pi(z+m)}{(z+1)(z+2)\cdots(z+m)}.$$

It is the analytic continuation of $\Pi(z)$ to $\mathrm{Re}(z) > -m-1$ except at $z = -1, -2, \dots, -m$.

5. Meromorphic functions

Book: pages 71 to 93

We considered holomorphic functions on open sets until now. We start to analyze functions with singularities. In this chapter, we always consider *isolated* singularities. That is, there exists a disc around a singularity such that the function is holomorphic everywhere is that disk except at the singularity.

5.1. Poles and zeros.

Notation. A *punctured disk* is the set given by a disk except its centered, denoted by $\dot{D}(z_0, r) := D(z_0, r)\backslash\{z_0\}$.

We have three types of isolated singularities:

Definition 5.1. Let f be a holomorphic function with a singularity at z_0 and $D(z_0, r)$ a disk such that f is defined on $D(z_0, r)\backslash\{z_0\}$. Then the point z_0 is

(1) a *removable singularity* if f is bounded on $\dot{D}(z_0, r)$,
(2) a *pole* if $f(z) \to \infty$ when $z \to z_0$,
(3) an *essential singularity* otherwise.

Example 5.2.

(1) $\frac{\sin(z)}{z}$ has a removable singularity at $z_0 = 0$.
(2) $\frac{1}{z}$ has a pole at $z_0 = 0$.
(3) $e^{1/z}$ has an essential singularity at $z_0 = 0$. We have

$$\lim_{x\to 0^-} e^{1/x} = 0, \qquad \lim_{x\to 0^+} e^{1/x} = \infty.$$

If $ix \to 0$ on the imaginary axis, the function $e^{1/(ix)}$ oscillates and does not converges.

Proposition 5.3 (Riemann)**.** *Let f be a holomorphic function on $\dot{D}(z_0, r)$ with a removable singularity at z_0. Then f has an analytic continuation to $D(z_0, r)$.*

Proof. In $D(z_0, r)$, define

$$g(z) = \begin{cases} (z-z_0)^2 f(z) & \text{if } z \neq z_0, \\ 0 & \text{if } z = z_0 \end{cases}.$$

Note that g is holomorphic on $\dot{D}(z_0, r)$. We show that it is also holomorphic at z_0. Since f is bounded, $\lim_{h\to 0} hf(z_0 + h) = 0$. Then

$$g'(z_0) = \lim_{h\to 0} \frac{g(z_0+h) - g(z_0)}{h} = \lim_{h\to 0} \frac{h^2 f(z+h) - 0}{h} = 0.$$

Therefore g is a holomorphic function on $D(z_0, r)$. It admits a Taylor series

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

Since $g(z_0) = g'(z_0) = 0$, we have $b_0 = b_1 = 0$. Then

$$F(z) = \frac{g(z)}{(z - z_0)^2} = \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2}$$

is an analytic continuation of f to $D(z_0, r)$. □

Definition 5.4. Let f be a holomorphic function that is not the zero function. A point $z_0 \in \mathbb{C}$ is a *zero* of f if $f(z_0) = 0$. The *order* of the zero is the smallest n such that $f^{(n)}(z_0) \neq 0$. If $n = 1$, the zero is *simple.*

Proposition 5.5. *Let f be a holomorphic function with a zero at z_0 that is not the zero function. Then there is a function g, $r > 0$ and a unique integer n such that g does not vanish on $D(z_0, r)$ and*

$$f(z) = (z - z_0)^n g(z)$$

for $z \in D(z_0, r)$. Note that n is equal to the degree of the zero and that f does not vanish on $\dot{D}(z_0, r)$.

Proof. Since f is not the zero function, there is $r > 0$ such that $f(z) \neq 0$ in $\dot{D}(z_0, r)$ by the identity theorem (Theorem 4.8 in Chapter 2 of the book). The Taylor series in $D(z_0, r)$ is

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}$$

with $a_n \neq 0$. Since the Taylor coefficients are given by $a_k = \frac{f^{(k)}(z_0)}{k!}$, n corresponds to the order of the zero. That last series defines $g(z)$. It does not vanish on $D(z_0, r)$ since f does not vanish on the punctured disk and $g(z_0) = a_n \neq 0$.

If there is another function h that does not vanish on $D(z_0, r)$ and another integer m such that $f(z) = (z - z_0)^m h(z)$, then

$$g(z) = (z - z_0)^{n-m} h(z).$$

If $n > m$, then $g(z_0) = 0$. If $n < m$, then $h(z_0) = 0$. In both cases, we get a contradiction. □

Proposition 5.6. *Let f be a holomorphic function that does not vanish on $\dot{D}(z_0, R)$ and with a pole at z_0. Then there is a function g and a unique integer n such that, for all $0 < r < R$, g does not vanish on $D(z_0, r)$ and*

$$f(z) = (z - z_0)^{-n} g(z)$$

for $z \in \dot{D}(z_0, r)$. Moreover

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z) \tag{5.1}$$

for a holomorphic function G.

Proof. Since f does not vanish on $\dot{D}(z_0, R)$, the function $h = \frac{1}{f}$ is holomorphic and bounded on $\dot{D}(z_0, r)$ for all $0 < r < R$ (it is unbounded on $\dot{D}(z_0, R)$ if f has a pole on the boundary). By the previous theorem $h(z) = (z - z_0)^n \tilde{g}(z)$ for some non-vanishing function $\tilde{g}$ and integer n. Then $g = \frac{1}{\tilde{g}}$ is a holomorphic function on $D(z_0, r)$ and

$$f(z) = (z - z_0)^{-n} g(z).$$

The integer n is unique for the same reasons as in the previous proof. We can write $g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ with $b_0 \neq 0$ since $g(z_0) \neq 0$. Then $f(z)$ is given by the formula in the theorem with $a_{-k} = b_{n-k}$ and

$$G(z) = \sum_{k=n}^{\infty} b_k (z - z_0)^{k-n}.$$

□

Definition 5.7. The integer n in the above proposition is the *order* of the pole. If $n = 1$, the pole is *simple*. The coefficient a_{-1} is the *residue* of f at z_0, denoted by $\operatorname{res}_{z_0} f(z) := a_{-1}$. The principal part of f is

$$f(z) - G(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}.$$

When f has a simple pole at z_0, it is clear that

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

More generally, the formula is the following.

Proposition 5.8. *Let f be a function with a pole of order n at z_0. Then*

$$\operatorname{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \Big[(z - z_0)^n f(z) \Big].$$

Proof. This is a direct consequence of Equation (5.1). We have

$$(z - z_0)^n f(z) = a_{-n} + a_{-n+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{n-1} + G(z)(z - z_0)^n.$$

Differentiating $n - 1$ times leaves the term with the residue and the term

$$\frac{d^{n-1}}{dz^{n-1}} G(z)(z - z_0)^n$$

that vanishes at z_0. □

Proposition 5.9 (Casorati-Weierstrass theorem)**.** *Let f be a function on $\dot{D}(z_0, R)$ with an essential singularity at z_0. Then the set*

$$\{f(z) : z \in \dot{D}(z_0, r)\}$$

is dense in $\mathbb{C}$ for all $0 < r < R$.

Proof. By contradiction, suppose that there is $w \in \mathbb{C}$ and $\rho > 0$ such that $D(w, \rho)$ and $f(\dot{D}(z_0, r))$ are disjoint. Then the function

$$g(z) = \frac{1}{f(z) - w}$$

is a function on $\dot{D}(z_0, r)$ that is bounded by ρ. So z_0 is a removable singularity for g and it admits an analytic extension $G(z)$ to $D(z_0, r)$. Independently of the value of G at z_0, G does not vanish on a small disk of radius r' around z_0. Then

$$f(z) = \frac{1}{G(z)} + w$$

on $\dot{D}(z_0, r')$. Either $G(z_0) = 0$ and $f(z) \to \infty$ as $z \to z_0$, or $G(z_0) \neq 0$ and the above formula gives an analytic continuation of f to the point z_0. In the first case, z_0 is a pole of f. In the second case, z_0 is a removable singularity of f. In both cases we have a contradiction. □

Theorem 5.10 (Picard's great theorem)**.** *Let f be a function on $\dot{D}(z_0, R)$ with an essential singularity at z_0. Then the set*

$$\{f(z) : z \in \dot{D}(z_0, r)\}$$

misses at most one complex number.

5.2. **Laurent series.**

Definition 5.11. Let $z_0 \in \mathbb{C}$ and $0 < R_1 < R_2 \leq \infty$. The open *annulus* centered at z_0 of radii R_1 and R_2 is

$$A(z_0, R_1, R_2) := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}.$$

Note that $\dot{D}(z_0, r) = A(z_0, 0, r)$ and $A(z_0, R_1, R_2) = \emptyset$ if $R_1 \geq R_2$.

Definition 5.12. The *Laurent series* centered at $z_0 \in \mathbb{C}$ given by $(a_n)_{n=-\infty}^{\infty}$ is

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We say that it converges if the two series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

converge. The second series is the *principal part* and the coefficient a_{-1} is the residue of the series.

We make sense of that infinite sum using the following result.

Lemma 5.13. *Let* $(a_n)_{n=-\infty}^{\infty}$ *and*

$$R_1 = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}, \qquad R_2 = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

The Laurent series centered at $z_0 \in \mathbb{C}$ *and given by* $(a_n)_{n=-\infty}^{\infty}$ *converges absolutely and uniformly on compact subsets of* $A(z_0, R_1, R_2)$*. More precisely, the two series*

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} \quad \text{and} \quad \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converge absolutely and uniformly on compact.

Proof. Let $f(z) = f_-(z) + f_+(z)$ with

$$f_-(z) = \sum_{n\leq -1} a_n (z - z_0)^n, \qquad f_+(z) = \sum_{n\geq 0} a_n (z - z_0)^n.$$

We know that f_+ converges absolutely and uniformly on compacts in the disk $D(z_0, R_2)$ by Theorem 2.7. Let

$$g(w) = \sum_{n>0} a_{-n} w^n.$$

Then $f_-(z) = g(w)$ for $w = \frac{1}{z-z_0}$ and $z \neq z_0 \Leftrightarrow w \neq 0$. We see that $g(w)$ converges in $D(0, R_1^{-1})$. If $|w| < R_1^{-1}$, then for $z = \frac{1}{w} + z_0$, we have

$$|z - z_0| = \frac{1}{|w|} > R_1$$

Therefore $f_-(z)$ converges in $A(z_0, R_1, \infty)$. Taking the intersection of the two domains of convergence, we see that f converges absolutely in $A(z_0, R_1, R_2)$ and uniformly on compact subset. □

Remark. In particular, the Laurent series defines a holomorphic function in $A(z_0, R_1, R_2)$. Its derivative can be obtained by termwise differentation thanks to Theorems 5.2 and 5.3 in Chapter 2 of the book:

$$\left(\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n\right)' = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1}.$$

Observe that the term of degree -1 vanishes.

Theorem 5.14. *Let f be a holomorphic function on $A(z_0, R_1, R_2)$. Then f admits a Laurent series:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in A(z_0, R_1, R_2)$. The coefficients are given by the Cauchy-like formula:

$$a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}} dw \tag{5.2}$$

for $R_1 < r < R_2$.

Proof. First, note that Equation (5.2) does not depend on the radius r by Theorem 3.6. This is because $\frac{f(w)}{(w-z_0)^{n+1}}$ is holomorphic in $A(z_0, R_1, R_2)$. We consider the sequence $(a_n)_{n=-\infty}^{\infty}$ given by Equation (5.2). We want to show that

$$g(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converges in $A(z_0, R_1, R_2)$ and is equal to $f(z)$. Let $z \in A(z_0, R_1, R_2)$ be fixed and $0 < \delta < \min\{|z - z_0| - R_1, R_2 - |z - z_0|\}$. So $\bar{D}(z, \delta) \subseteq A(z_0, R_1, R_2)$ and

$$f(z) = \frac{1}{2\pi i} \int_{|w-z|=\delta} \frac{f(w)}{w - z} dw.$$

Consider the curve given by the w in the 3 circles $|w - z| = \delta$, $|w - z_0| = |z - z_0| - \delta =: r_1$ and $|w - z_0| = |z - z_0| + \delta =: r_2$ (Do a drawing!). The circle of radius r_1 is traveled in the positive orientation, the circle of radius r_2 is traveled in the negative orientation and we branch at the two intersection points with the circle $|w - z| = \delta$.

Claim: γ is freely homotopic to a constant curve within $A(z_0, R_1, R_2)\backslash\{z\}$ (no proof).

By Cauchy's theorem:

$$0 = \int_{|w-z|=\delta} \frac{f(w)}{w-z} dw + \int_{|w-z_0|=r_1} \frac{f(w)}{w-z} dw - \int_{|w-z_0|=r_2} \frac{f(w)}{w-z} dw.$$

We compute:

$$\begin{aligned}
2\pi i f(z) &= \int_{|w-z|=\delta} \frac{f(w)}{w-z} dw \\
&= \int_{|w-z_0|=r_2} \frac{f(w)}{w-z} dw - \int_{|w-z_0|=r_1} \frac{f(w)}{w-z} dw \\
&= \int_{|w-z_0|=r_2} \frac{f(w)}{w-z_0} \frac{1}{1 - \frac{z-z_0}{w-z_0}} dw - \int_{|w-z_0|=r_1} \frac{f(w)}{z-z_0} \frac{1}{\frac{w-z_0}{z-z_0} - 1} dw. \\
&= \int_{|w-z_0|=r_2} \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n dw + \int_{|w-z_0|=r_1} \frac{f(w)}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n dw
\end{aligned}$$

The two geometric series converge uniformly for w on their respective domain. Therefore, we can exchange sum and integral. We get

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\int_{|w-z_0|=r_2} \frac{f(w)}{(w-z_0)^{n+1}} dw \right) (z-z_0)^n + \sum_{n=0}^{\infty} \left(\int_{|w-z_0|=r_1} \frac{f(w)}{(w-z_0)^{-n}} dw \right) (z-z_0)^{-(n+1)} \\
&= 2\pi i \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.
\end{aligned}$$

So the function f is equal to the Laurent series g. ∎

Remark. Like Taylor series, Laurent series are unique.

Theorem 5.15 (Uniqueness theorem for Laurent series)**.** *Let f and g be two holomorphic functions on a region $M \subseteq \mathbb{C}$ with an isolated singularity at z_0 that is a pole or a removable singularity. Suppose that on a sequence $z_n \neq z_0$ that converges to z_0, we have $f(z_n) = g(z_n)$. Then $f(z) = g(z)$ on M.*

Remarks.

(1) If f and g are the same of a sequence that converge to a point of M, we can of course use the uniqueness theorem for holomorphic functions.
(2) This is not true for essential singularities. Consider $f(z) = e^{1/z}$, $g(z) = 1$ and $z_n = \frac{1}{2\pi i n} \to 0$. Then
$$f(z_n) = 1 = g(z_n).$$

Proof. We write the Laurent series of f and g at z_0:
$$f(z) = \sum_{n=N_1} a_n(z - z_0)^n, \qquad g(z) = \sum_{n=N_2} b_n(z - z_0)^n$$
for $z \in D(z_0, r)$. Let $N = \min\{N_1, N_2\}$. Then
$$F(z) = (z - z_0)^{-N} f(z) \text{ and } G(z) = (z - z_0)^{-N} g(z)$$
are functions on $D(z_0, r)$ with a removable singularity at z_0 and such that $G(z_n) = F(z_n)$. Applying the uniqueness theorem to their analytic extension, we see that $F(z) = G(z)$. Then $f(z) = g(z)$ on $\dot{D}(z_0, r)$. We take a sequence in $\dot{D}(z_0, r)$ with limit in the punctured disk and apply the uniqueness theorem for holomorphic function to deduce that $f(z) = g(z)$ on M. □

Proposition 5.16. *Let f be a holomorphic function on $\dot{D}(z_0, r)$ with a Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$. Then*

(1) *f has a removable singularity at z_0 if and only if $a_n = 0$ for $n < 0$.*
(2) *f has a pole at z_0 if and only if there is $N \leq 0$ such that $a_n = 0$ for $n < N$.*
(3) *f has an essential singularity at z_0 if and only if $a_n \neq 0$ for infinitely many negative n.*

Proof. We will use the uniqueness of Laurent series multiple times.

(1) f has a removable singularity $\Leftrightarrow$ f can be extended to a holomorphic function on $D(z_0, r)$ $\Leftrightarrow$ f has a Taylor series centered at z_0 $\Leftrightarrow$ $a_n = 0$ for n negative.
(2) One direction is given by Proposition 5.6. The other direction is clear.
(3) Since the other cases are equivalences and the three cases are mutually exclusive, we can also conclude for that part.

□

Remark. In (2), the largest negative n such that $a_n \neq 0$ is the order of the pole. If f is holomorphic at z_0 or has a removable singularity, the smallest positive n such that $a_n \neq 0$ is the order of the zero at z_0.

5.3. The residue theorem.

Definition 5.17. Let γ be a closed curve and $z \in \mathbb{C}$ not on γ. The *winding number* of γ around z is
$$W_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

See also the book: page 347.

Example 5.18. Let γ be the circle $|z - z_0| = r$ parametrized by $u(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then
$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{u'(t)}{u(t) - z_0} dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dt = 1.$$

Proposition 5.19. *Let γ be a closed curve in $\mathbb{C}$.*

(1) *If $z \notin \gamma$, then $W_\gamma(z) \in \mathbb{Z}$.*

(2) *If w and z belong to the same open connected component in the complement of γ, then $W_\gamma(w) = W_\gamma(z)$.*
(3) *If z belongs to the unbounded connected component in the complement of γ, then $W_\gamma(z) = 0$.*
(4) *If γ is simple, then $W_\gamma(z) \in \{-1, 0, 1\}$ for all $z \in \mathbb{C}$.*
(5) *The closed curves γ_0 and γ_1 are freely homotopic to each other within $\mathbb{C}\backslash\{z\}$ if and only if $W_{\gamma_0}(z) = W_{\gamma_1}(z)$.*

Proof.

(1) Parametrize γ as $z + u(t)$ with $u : [0, 1] \to \mathbb{C}\backslash\{0\}$. Let
$$v(t) := \int_0^t \frac{u'(s)}{u(s)} ds + \log(u(0)).$$
Then $\frac{1}{2\pi i}(v(1) - v(0))$ is the winding number of u around z. Also v is continuous and its derivative is $\frac{u'(t)}{u(t)}$. Then
$$(ue^{-v})' = u'e^{-v} - uv'e^{-v} = 0$$
So $e^v = cu$ for some constant c. Since $e^{v(0)} = u(0)$, $c = 1$. Taking $t = 0$, we get
$$e^{v(0)} = u(0) = u(1) = e^{v(1)}$$
since u is closed. Therefore $2\pi i W_\gamma(z) = v(1) = 2\pi i k$ for some $k \in \mathbb{Z}$.

(2) Since $W_\gamma(z)$ is a continuous function and it is integer valued, it must be constant on connected component.

(3) Note that
$$\lim_{z\to\infty} W_\gamma(z) = \lim_{z\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} = 0$$
since the denominator goes to infinity. Applying part (2), we conclude.

(4) (Sketch) By contraposition, if the winding number is $k > 1$, then the imaginary part of v has a range of diameter $\geq 2\pi i k$. The real part of v is the same at 0 and at 1. By the intermediate value theorem, there is $t_0 \neq t_1$ such that $v(t_1) - v(t_0) = 2\pi i$. Then $u(t_0) = u(t_1)$ and γ is not simple.

(5) If γ_0 and γ_1 are freely homotopic to each other within $\mathbb{C}\backslash\{z\}$, then $W_{\gamma_0}(z) = W_{\gamma_1}(z)$ by the free homotopy theorem 3.6. Conversely, let γ_0 and γ_1 be parametrized by $z + u_0$ respectively $z + u_1$. We can define v_0 and v_1 as in the proof of (1) such that $e^{v_0} = u_0$ and $e^{v_1} = u_1$. Define $v(\tau, t) := (1-\tau)v_0(t)+\tau v_1(t)$. Then $\gamma(\tau, t) := z+e^{v(\tau,t)}$ is a free homotopy of curves between γ_0 and γ_1 within $\mathbb{C}\backslash\{z\}$. Clearly $u(\tau, t)$ is never equal to z since the exponential does not vanish. Moreover $v(\tau, 1) - v(\tau, 0)$ is the same multiple of $2\pi i$ as the common value of $v_0(1) - v_0(0)$ and $v_1(1) - v_1(0)$. So it is a closed curve for all τ.

□

Example 5.20. Let γ be the circle $|z - z_0| = r$. From point (2) and (3), we have
$$W_\gamma(z) = \begin{cases} 1 & \text{if } |z - z_0| < r, \\ 0 & \text{if } |z| > 1. \end{cases}$$

Theorem 5.21 (Residue theorem). *Let $M \subseteq \mathbb{C}$ be open and γ a closed curve in M that is contractible within M (freely homotopic to the constant curve). Let f be a holomorphic function in M except for isolated singularities at the points $z_1, \dots, z_N$ that are not on γ. Then*
$$\int_\gamma f(z)dz = 2\pi i \sum_{n=1}^N W_\gamma(z_n) \operatorname{res}_{z_n} f.$$

Proof. Since the number of singularities is finite, there is $r > 0$ such that $\dot{D}(z_n, r) \subseteq M$ for all n and such that all these disks are disjoints. Let n be fixed. The function f has a Laurent series around z_n:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k^{(n)} (z - z_n)^k$$

for $z \in \dot{D}(z_n, r)$. We write the Laurent series as $f_n^-(z) + f_n^+(z)$ with f_n^- the principal part. Note that f_n^- converges on $\mathbb{C}\backslash\{z_n\}$ and f_n^+ converges on $D(z_n, r)$. Let

$$g(z) := f(z) - \sum_{n=1}^{N} f_n^-(z).$$

Then g is holomorphic on M and each z_n is a removable singularity. More precisely, on $D(z_n, r)$, we have

$$g(z) = f_n^+(z) + f_n^-(z) - \sum_{m=1}^{N} f_m^-(z) = f_n^+(z) - \sum_{\substack{m=1 \\ m \neq n}}^{\infty} f_m^-(z)$$

and each term on the right converges on $D(z_n, r)$. So $g(z)$ has a holomorphic extension to M. By the free homotopy theorem 3.6 $\int_\gamma g = 0$. Therefore

$$\int_\gamma f = \sum_{n=1}^{N} \int_\gamma f_n^-.$$

Then

$$\int_\gamma f_n^- = \int_\gamma \frac{a_{-1}}{w - z_n} dw + \int_\gamma \sum_{k \le -2} a_k (w - z_n)^k dw.$$

The last sum converges uniformly and has a primitive

$$\sum_{k \le -2} a_k \frac{(w - z_n)^{k+1}}{k+1}.$$

Hence

$$\int_\gamma f_n^- = \int_\gamma \frac{a_{-1}}{w - z_n} dw = a_{-1} \cdot 2\pi i W_\gamma(z_n) = 2\pi i W_\gamma(z_n) \operatorname{res}_{z_n} f.$$

□

Examples 5.22. See the book: Section 2.1 in Chapter 3.

Example 5.23. The Basel problem (solved by Euler in 1734) asks for the value of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This can be computed using the residue theorem. More generally, consider

$$f(z) = \pi \cot(\pi z)$$

defined by

$$\pi \cot(\pi z) = \frac{\pi}{\tan(\pi z)} = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \left(1 + \frac{2}{e^{2\pi i z} - 1}\right).$$

Note that $f(z)$ has a pole if and only if$e^{2\pi i z} = 1 \Leftrightarrow z = k,\ k \in \mathbb{Z}$. By periodicity, we can compute the residue only at 0:

$$\operatorname{res}_k f(z) = \operatorname{res}_0 f(z) = \pi i \lim_{z \to 0} \left(1 + \frac{2}{e^{2\pi i z} - 1}\right) z = 2\pi i \lim_{z \to 0} \frac{z}{e^{2\pi i z} - 1} = 2\pi i \lim_{z \to 0} \frac{1}{2\pi i e^{2\pi i z}} = \frac{2\pi i}{2\pi i} = 1.$$

Consider the contour γ_R given by the boundary of the rectangle $[-R - \frac{1}{2}, R + \frac{1}{2}]^2$ for R a positive integer. The function f is uniformly bounded on γ_R: since $\left|e^{2\pi i z} - 1\right| \ge \frac{1}{2}$, we have $|f(z)| \le 5\pi$.

Let g be a holomorphic function on $\mathbb{C}\backslash\{0\}$ with $|g(z)| = O(z^{-1-\epsilon})$ for some $\epsilon > 0$. For example, $g(z) = z^{-2}$. By the residue theorem:

$$\int_{\gamma_R} f(z)g(z)dz = 2\pi i \operatorname{res}_0(fg) + 2\pi i \sum_{\substack{n=-R \\ n\neq 0}}^{R} g(n).$$

Moreover

$$\left|\int_{\gamma_R} f(z)g(z)dz\right| \leq 4(2R+1)O(R^{-1-\epsilon}) = O(R^{-\epsilon}).$$

If $R \to \infty$, we get

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} g(n) = -\operatorname{res}_0(fg).$$

It remains to compute the residue at 0. We do that for $g(z) = z^{-2}$. We need to compute the Laurent series of

$$\frac{\pi\cot(\pi z)}{z^2} = \pi i\left(\frac{1}{z^2} + \frac{2}{z^2(e^{2\pi i z} - 1)}\right).$$

First, consider the function $\frac{z}{e^z-1}$. It is a holomorphic function with value 1 at 0. We compute its Taylor series in the following way:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} a_n z^n \qquad \Leftrightarrow \qquad z = (a_0 + a_1 + a_2 + \dots)\left(z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right).$$

Identifying the terms on the left and on the right, we get the following equations:

$$\begin{aligned} z &= a_0 z, \\ 0 &= \frac{a_0}{2}z^2 + a_1 z^2, \\ 0 &= \frac{a_0}{6}z^3 + \frac{a_1}{2}z^3 + a_2 z^3. \end{aligned}$$

Solving one equation after the other, we get

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{12}.$$

Then we get

$$\frac{\pi\cot(\pi z)}{z^2} = \pi i\left(\frac{1}{z^2} + \frac{2}{2\pi i z^3}\left(1 - \frac{2\pi i z}{2} + \frac{(2\pi i z)^2}{12} + \dots\right)\right) = \frac{1}{z^3} - \frac{\pi^2}{3z}.$$

Finally, $\operatorname{res}_0(fg) = \frac{\pi^2}{3}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

More generally, the coefficients a_n of the Taylor series of $\frac{z}{e^z-1}$ are linked to the Bernoulli numbers, denoted B_n. We have

$$a_n = \frac{B_n}{n!} \qquad \Leftrightarrow \qquad B_n = \frac{d^n}{dz^n}\left(\frac{z}{e^z-1}\right).$$

They are 0 if n is odd and larger than 1. The same proof shows that

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{1}{2}(2\pi i)^{2k}\frac{B_{2k}}{(2k)!} = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}.$$

Note that this argument does not work for odd values of zeta since

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^{2k+1}} = \zeta(2k+1) - \zeta(2k+1).$$

Moreover $B_{2k+1} = 0$.

5.4. **Meromorphic functions.**

Definition 5.24. Let f be a holomorphic function with a singularity at z_0. Let a_n, $n \in \mathbb{Z}$, denotes the n-th coefficient in the Laurent series of f at z_0. The *valuation* of f at z_0 is

$$v_{z_0}(f) = \min\{n \in \mathbb{Z} : a_n \neq 0\}.$$

Example 5.25. If z_0 is a removable singularity, $v_{z_0}(f) \geq 0$. If z_0 is a pole, $v_{z_0}(f) < 0$. If z_0 is an essential singularity, $v_{z_0}(f) = -\infty$. Note that $|v_{z_0}(f)|$ is the order of the pole or the zero of f at z_0.

Theorem 5.26. *Let $f, g : \dot{D}(z_0, r) \to \mathbb{C}$ be holomorphic functions that are not identically 0. Assume f and g have a removable or a pole singularity at z_0. Then $f+g$, $f-g$, fg and f/g also have a removable or a pole singularity at z_0. Moreover*

1. $v_{z_0}(f \pm g) \geq \min\{v_{z_0}(f), v_{z_0}(g)\}$ *with equality if* $v_{z_0}(f) \neq v_{z_0}(g)$,
2. $v_{z_0}(fg) = v_{z_0}(f) + v_{z_0}(g)$,
3. $v_{z_0}(f/g) = v_{z_0}(f) - v_{z_0}(g)$.

Proof. Let $M = v_{z_0}(f)$ and $N = v_{z_0}(g)$ and

$$\sum_{m=M}^{\infty} a_m (z-z_0)^m, \qquad \sum_{n=N}^{\infty} a_n (z-z_0)^N$$

be the Laurent series of f resp. g.

1. This follows directly from the definitions, by adding the Laurent series of f and g. Note that if the smallest coefficient of the Laurent series is the same up to a sign, there might be cancellation, but only in that case. More precisely, the Laurent series of $f \pm g$ is

$$\sum_{k=\min\{M,N\}} (a_k \pm b_k)(z-z_0)^k.$$

   If $M < N$, then $b_M = 0$ and there is no cancellation at the smallest coefficient. Same if $M > N$.
2. Multiplying two absolute convergent series gives an absolute convergent series:

$$\left(\sum_{m=M}^{\infty} a_m (z-z_0)^m\right)\left(\sum_{n=N}^{\infty} a_n (z-z_0)^N\right) = \sum_{m=M}^{\infty}\sum_{n=N}^{\infty} a_m b_n (z-z_0)^{m+n}.$$

   The term with the smallest power is

$$a_M b_N (z-z_0)^{M+N}.$$

   Hence the result.
3. From Propositions 5.5 and 5.6, we have that

$$f(z) = (z-z_0)^M \tilde{f}(z), \qquad g(z) = (z-z_0)^N \tilde{g}(z)$$

   for some functions $\tilde{f}, \tilde{g}$ that don't vanish on $\dot{D}(z_0, r')$ for some $0 < r' \leq r$. Then

$$\frac{f(z)}{g(z)} = (z-z_0)^{M-N} \frac{\tilde{f}(z)}{\tilde{g}(z)}.$$

   The function $\frac{\tilde{f}}{\tilde{g}}$ is a holomorphic and non-vanishing function on $\dot{D}(z_0, r/2)$. By uniqueness of the power in $(z-z_0)$ in the above Propositions, we deduce that $v_{z_0}(f/g) = M - N$.

□

Example 5.27. In the above theorem, if f or g has an essential singularity, then (1) and (2) are still valid but (3) breaks: $(e^{1/z}-1)^{-1}$ has a sequence of poles that converge to 0, given by $\frac{1}{2\pi ik}$, $k \in \mathbb{Z}$. So the singularity is not isolated anymore.

Definition 5.28. A set $S \subseteq \mathbb{C}$ is *discrete* if every intersection of S with a compact is finite.

Definition 5.29. Let $M \subseteq \mathbb{C}$ be an open set. A function f is *meromorphic* on M if every singularity of f on M is in a discrete set S and is a removable singularity or a pole.

Let f be a function that is holomorphic in a neighborhood of infinity, that is in the complement of a disk $\bar{D}(0,R)$ for R large enough. It is *meromorphic at infinity* if $z \mapsto f(1/z)$ is meromorphic at 0.

If two meromorphic functions f and g on M differ only by their removable singularities, we should identify them.

Example 5.30. The functions $\frac{z}{z}$ and 1 are identified together. The functions 1 on $\mathbb{C}$ and 1 on $\mathbb{C}\backslash\{0\}$ also.

Theorem 5.31. *Let $M \subseteq \mathbb{C}$ be a region. The set of meromorphic functions on M is a field, called the* function field *or the* global field *of M.*

Proof. Let f and g be non-identically zero meromorphic functions on M with singularities at S_f resp. S_g. By Theorem 5.26, we know that $f \pm g$ and fg have singularities at $S_f \cup S_g$ which is also discrete. So $f \pm g$ and fg are also meromorphic functions. The set of singularities of f/g is given by $S_f \cup S_g \cup Z_g$, where Z_g is the set of zeros of g. By the uniqueness theorem, Z_g is also discrete. So f/g is a meromorphic function. □

Theorem 5.32. *The set of functions that are meromorphic on $\mathbb{C}$ and at infinity is exactly the set of rational functions*

$$\left\{\frac{P}{Q} : P, Q \in \mathbb{C}[z],\ Q \neq 0\right\}.$$

Proof. See the book, Theorem 3.4 in Chapter 3. □

Remark. In particular, rational functions are determined up to a multiplicative constant by the locations and the valuations of their singularities.

Riemann sphere: see the book, pages 88 and 89.

Proposition 5.33. *Let $M \subseteq \mathbb{C}$ be open and f a meromorphic function on M. Then f has an unique extension to a continuous function $F : M \to \mathbb{C} \cup \{\infty\}$.*

Remark. In the language of Riemann surface, F is a holomorphic function from M to the Riemann sphere. Actually, the set of meromorphic functions on M is exactly the set of holomorphic functions from M to $\mathbb{C} \cup \{\infty\}$. By Theorem 5.32, the set of holomorphic functions functions from the Riemann sphere to itself is the set of rational functions.

Proof. Clearly, at a singularity z_0 of f, we have to define $F(z_0)$ by $\lim_{z\to z_0} f(z)$. If z_0 is a removable singularity, then this is finite and we know that we have a holomorphic extension to z_0. If z_0 is a pole, then $|f(z)| \to \infty$ when $z \to z_0$, so $F(z_0) = \infty$. □

5.5. **The argument principle and applications.** Let $M \subseteq \mathbb{C}$ be a region and $\gamma \subseteq M$ be a closed curve that is contractible in M. Let f be a meromorphic function on M with no zero or pole on γ. Then the image $f(\gamma)$ is a closed curve in $\mathbb{C}\backslash\{0\}$, so it has some winding number around 0. Let $u : [a,b] \to M$ be a parametrization of γ. By definition

$$2\pi i W_{f(\gamma)}(0) = \int_{f(\gamma)} \frac{dw}{w} = \int_a^b \frac{(f(u(t)))'}{f(u(t))} dt = \int_a^b \frac{f'(u(t))}{f(u(t))} u'(t) dt = \int_\gamma \frac{f'}{f}.$$

Proposition 5.34. *The derivative of a meromorphic function f is meromorphic. Moreover if f is not the constant function 0, then $\frac{f'}{f}$ is meromorphic and*

$$\operatorname{res}_{z_0} \frac{f'}{f} = v_{z_0}(f).$$

In particular, z_0 is a pole of $\frac{f'}{f}$ if and only if z_0 is a pole or a zero of f and it is always a simple pole.

Proof. Clearly, f' is holomorphic where f is holomorphic. If z_0 is a singularity of f, then it has a Laurent series with a finite principal part. Differentiating term by term, we see that f' has a Laurent series with a finite principal part as well. So the singularities of f' are a subset of the singularities of f and they are all poles or removable singularities. Therefore f' is meromorphic. Since f has a discrete set of zeros, $\frac{f'}{f}$ is also meromorphic. Moreover, we can write

$$f(z_0) = (z - z_0)^N g(z)$$

with $N = v_{z_0}(f)$ and $g(z)$ a non-vanishing holomorphic function in a neighborhood of z_0. Then

$$\frac{f'(z_0)}{f(z_0)} = \frac{N(z-z_0)^{N-1}g(z) + (z-z_0)^N g'(z_0)}{(z-z_0)^N g(z)} = \frac{N}{z - z_0} + \frac{g'(z_0)}{g(z_0)}.$$

The second term is holomorphic since g does not vanish. Then clearly

$$\operatorname{res}_{z_0} \frac{f'}{f} = v_{z_0}(f).$$

□

Remark. Recall also that $\log(f(z))$ is a multiple-valued function and $\log(fg) \neq \log(f) + \log(g)$ in general. The situation is way better for its derivative $\frac{f'}{f}$. It is a well defined meromorphic function and

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}.$$

It even generalizes to

$$\frac{\left(\prod_{n=1}^N f_n\right)'}{\prod_{n=1}^N f_n} = \sum_{n=1}^N \frac{f'}{f}.$$

Theorem 5.35 (Argument principle)**.** *Let $M \subseteq \mathbb{C}$ be a region and $\gamma \subseteq M$ be a closed curve that is contractible inside M. Let f be a meromorphic function on M with no zero or pole on γ. Then*

$$\int_\gamma \frac{f'}{f} = W_{f(\gamma)}(0) = \sum_{z_0} W_\gamma(z_0) v_{z_0} f$$

where the sum runs over the finitely many zeros and poles of f for which $W_\gamma(z_0) \neq 0$. In particular, if γ is simple and positively oriented, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = W_{f(\gamma)}(0) = (\text{number of zeros of } f \text{ inside } \gamma) - (\text{number of poles of } f \text{ inside } \gamma)$$

both counted with multiplicity.

Remark. Jordan's curve theorem says that for a simple curve γ, the set $\mathbb{C}\backslash\gamma$ has two connected component. One of them is bounded and called the *interior* of γ. The other one is unbounded. A simple curve is contractible inside M if and only if its interior lies inside M.

Proof. By the residue theorem and the Proposition above, we have

$$\int_\gamma \frac{f'}{f} = 2\pi i \sum_{z_0} W_\gamma(z_0) \operatorname{res}_{z_0} \frac{f'}{f} = 2\pi i \sum_{z_0} W_\gamma(z_0) \operatorname{res}_{z_0} v_{z_0}(f).$$

□

Theorem 5.36 (Rouché)**.** *Let $M \subseteq \mathbb{C}$ be a region and $\gamma \subseteq M$ be a simple closed curve that is contractible inside M. Let f and g be two holomorphic functions on M with*

$$|f(z)| > |g(z)|$$

for $z \in \gamma$. Then f and $f+g$ have the same number of zeros inside γ, counted with multiplicity.

Proof. First proof: see the book, Theorem 4.3 in Chapter 3.

Second proof: let S be the set of zeros of f and $f+g$ inside γ. We want to show that

$$\sum_{z_0 \in S} v_{z_0}(f) = \sum_{z_0 \in S} v_{z_0}(f+g).$$

By Theorem 5.26, this is equivalent to

$$0 = \sum_{z_0 \in S} v_{z_0}\left(\frac{f+g}{f}\right) = \sum_{z_0 \in S} v_{z_0}\left(1 + \frac{g}{f}\right).$$

Let $h = 1 + \frac{g}{f}$. By the argument principle, the above equation is equivalent to $W_{h(\gamma)}(0) = 0$. But for $z \in \gamma$, $|h(z) - 1| < 1$. So $h(\gamma) \subseteq D(1,1)$ and $W_{h(\gamma)}(0) = 0$. □

Corollary 5.37 (Fundamental theorem of algebra)**.** *A polynomial $P(z) \in \mathbb{C}[z]$ has exactly n roots counted with multiplicity.*

Proof. Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$. Let $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. Let $r > 1$ such that

$$r > \frac{|a_{n-1}| + \cdots + |a_1| + |a_0|}{a_n}.$$

Then if $|z| = r$, we have

$$\left|\frac{g(z)}{f(z)}\right| = \frac{1}{|a_n|}\left(\frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\right) \leq \frac{1}{|a_n|}\frac{|a_{n-1}| + \cdots + |a_1| + |a_0|}{r} < 1.$$

So on the circle $|z| = r$, $|f(z)| > |g(z)|$. By Rouché's theorem, f and $f + g = P$ have the same number of zeros inside $D(0,r)$. Clearly, f has one zero at 0 of multiplicity n. □

Theorem 5.38 (Open mapping theorem)**.** *A non-constant holomorphic map is open, that is the preimage of open sets are open sets.*

Proof. See the book, Theorem 4.4 in Chapter 3. □

Theorem 5.39 (Maximum modulus principle)**.** *Let f be a non-constant holomorphic function in a region M. Then f don't attain a maximum in M.*

Proof. See the book, Theorem 4.5 in Chapter 3. □

Remark. In particular, the maximum of a holomorphic function on a compact set is attained on its boundary. This is not true on a general closed set. The function $f(z) = e^{-iz^2}$ is unbounded on the set $\mathrm{Re}(z), \mathrm{Im}(z) \geq 0$. Consider the line $x = y$ where $f(z) = e^{y^2}$. But on the boundary lines x and iy, we have $|f(z)| = 1$.

Theorem 5.40 (Gauss-Lucas)**.** *Let $Q(z)$ be a non-constant complex polynomial. Every root of Q' lies in the convex hull of the roots of $Q(z)$. That is, if $z_1, \ldots, z_n$ are the roots of Q, then a root of Q' can be written as*

$$\sum_{k=1}^{n} \lambda_k z_k$$

with $\sum_{k=1}^{n} \lambda_k = 1$.

Proof. Let $Q(z) = a_n \prod_{k=1}^{n}(z - z_k)$. Then

$$\frac{Q'(z)}{Q(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k}.$$

Let w be a root of Q'. If $Q(w) = 0$, then we are done. If $Q(w) \neq 0$, then

$$\sum_{k=1}^{n} \frac{1}{w - z_k} = 0.$$

Taking the complex conjugate, we get

$$0 = \sum_{k=1}^{n} \frac{1}{\overline{w - z_k}} = \sum_{k=1}^{n} \frac{w - z_k}{|w - z_k|^2}.$$

Now write $\lambda_k = \frac{1}{|w - z_k|^2}$. Then

$$w = \frac{\sum_{k=1}^{n} \lambda_k z_k}{\sum_{k=1}^{n} \lambda_k}.$$

□

Theorem 5.41 (Bernstein)**.** *Let $P(z)$ be a non-constant complex polynomial of degree $n \geq 1$. Then*

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.$$

Equality holds if and only if $P(z) = az^n$.

Remark.

(1) Applying the theorem k times, we get

$$\max_{|z|=1} \left|P^{(k)}(z)\right| \leq \frac{n!}{(n-k)!} \max_{|z|=1} |P(z)|.$$

(2) By the maximum modulus principle, we can equivalently consider the maximums over $|z| \leq 1$.

Proof. Rescaling $P(z)$, we can suppose without loss of generality that $\max_{|z|=1} |P(z)| = 1$. Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Consider $\lambda z^n - P(z)$. By assumption $|\lambda z^n| > |P(z)|$ on the unit circle. By Rouché's theorem, $\lambda z^n - P(z)$ has the same number of zeros as λz^n in $D(0,1)$, which is n. So in fact, all the zeros of $\lambda z^n - P(z)$ are in $D(0,1)$. By the Gauss-Lucas theorem, the zeros of $n\lambda z^{n-1} - P'(z)$ also lie in $D(0,1)$. By contradiction, if $|P'(z_0)| > n$ for some z_0 with $|z_0| = 1$, then choose

$$\lambda = \frac{P'(z_0)}{n z_0^{n-1}}.$$

In that case, $|\lambda| > 1$ and $n\lambda z_0^{n-1} - P'(z_0) = 0$, contradicting that the zero of this polynomial are inside $D(0,1)$. In conclusion,

$$\max_{|z|=1} |P'(z)| \leq n = n \max_{|z|=1} |P(z)|.$$

□

Theorem 5.42. *Let f be a holomorphic function on a region $M \subseteq \mathbb{C}$. Suppose that f is injective and let $N = f(M)$. Then the inverse of f, $g : N \to M$, is holomorphic. Moreover*

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

Proof. By the open mapping theorem, for any open set $U \subseteq M$, we have that $g^{-1}(U) = f(U)$ is an open set. So g is continuous. We want to show that for any $z_0 \in N$, the limit

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

exists. Write $w = g(z)$ and $w_0 = g(z_0)$. Then $z = f(w)$ and $z_0 = f(w_0)$. Moreover, $z \to z_0$ implies that $w \to w_0$ by continuity. Then the above limits transform into

$$\lim_{w \to w_0} \frac{w - w_0}{f(w) - f(w_0)} = \frac{1}{f'(w_0)}.$$

It remains to show that $f'(w_0) \neq 0$. Note that if $f'(w_0) = 0$, then the Taylor series of f at w_0 is

$$f(w) = f(w_0) + \sum_{n=N}^{\infty} a_n (w - w_0)^n$$

with $N = v_{w_0}(f(w) - f(w_0))$. Then the function $\frac{f(w)-f(w_0)}{(w-w_0)^N}$ is non-zero around w_0 and admits a degree N roots. Hence

$$f(w) = f(w_0) + [(w - w_0)h(w)]^N.$$

Then the points $w_0 + re^{2\pi ik/N}$ have the same image, contradicting the injectivity of f. We conclude that

$$g'(z_0) = \frac{1}{f'(w_0)} = \frac{1}{f'(g(z_0))}.$$

Or the set $f'(w) = 0$ is discrete and so is its image. Then g is holomorphic everywhere except at these points and bounded there by continuity, so they are removable singularities. □

Proposition 5.43. *If f is entire and injective, then $f(z) = az + b$ for $a \neq 0$.*

Proof. First, if f is a polynomial of degree at least 2, then f has more than two roots. If the roots are not all the same, then 0 has multiple preimages, contradicting f being injective. If the roots are all the same, $f(z) = a(z - z_0)^n$ for $n \geq 2$. Then $f(z) = b$ has multiple solutions for $b \neq 0$. contradicting the hypothesis again. If f is a polynomial of degree 0, it is clearly not injective. If f is a polynomial of degree 2, this is the result we're looking for.

If f is not a polynomial, consider $f(1/z)$ which has an essential singularity at infinity. By Theorem 5.9, the image of $\{z \in \mathbb{C} : |z| > 1\}$ is dense in $\mathbb{C}$. By the open mapping theorem, the image of $D(0, 1)$ is open in $\mathbb{C}$. So there is $z_1, z_2 \in \mathbb{C}$ such that $|z_1| > 1$ and $|z_2 < 1|$ and $f(z_1) = f(z_2)$. So f is not injective. □

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