Decay of the Harish-Chandra inverse transform

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Let $h \leftrightarrow \kappa$ be two function related by the Harish-Chandra transform (see for example [Iwa02], Section 1.8). We prove a strong decay bound for $\kappa(u)$ when u is big enough. Let $h(\tau)$ be a function such that

$$T^j \frac{d^j}{d\tau^j} h(u) \ll_{A,j} \left(1 + \frac{|\tau|}{T}\right)^{-A}$$

and that satisfies the conditions to be inverted under the Harish-Chandra transform. Suppose moreover, that it is holomorphic in a strip with $\mathrm{Im}(\tau) \leq 2$. Then the geometric side of the pre-trace formula is negligible for $u \gg T^{-2}$. More precisely:

Lemma 0.1. *Let $\epsilon > 0$, $A > 0$, $T \geq 1$. Then*

$$\sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \\ u(z_T, \gamma z_T) \geq T^{-2+\epsilon}}} |\kappa(u(z_T, \gamma z_T))| \ll_{A,\epsilon} T^{-A}(\mathrm{Im}(z) + 1).$$

Remark. This proof was essentially done in [Fel23], Lemma 4.3.

Proof. We apply the usual three steps to get the Harish-Chandra inverse transform (see (1.64) in [Iwa02]). This gives

$$\begin{aligned}
g(r) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ir\tau} h(\tau) d\tau, \\
q(v) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} h(\tau)(\sqrt{v+1} + \sqrt{v})^{2i\tau} d\tau, \\
\kappa(u) &= \frac{1}{4\pi^2 i} \int_u^{\infty} \frac{1}{\sqrt{v-u}} \int_{-\infty}^{\infty} h(\tau) \frac{(\sqrt{v+1} + \sqrt{v})^{2i\tau}}{\sqrt{v(v+1)}} \tau d\tau \, dv.
\end{aligned}$$

We consider first $q(v)$. Since h is holomorphic in a strip, we can move the integration line to $\tau \mapsto \tau + 2i$:

$$\int_{-\infty}^{\infty} h(\tau)\tau(\sqrt{v+1} + \sqrt{v})^{2i\tau} d\tau = (\sqrt{v+1} + \sqrt{v})^{-4} \int_{-\infty}^{\infty} h(\tau + 2i)(\tau + 2i)(\sqrt{v+1} + \sqrt{v})^{2i\tau} d\tau.$$

Integrating by parts, we get

$$\begin{aligned}
4\pi q(v) &= (\sqrt{v+1}+\sqrt{v})^{-4}\int_{-\infty}^{\infty} h(\tau+2i)(\tau+2i)(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau \\
&= (\sqrt{v+1}+\sqrt{v})^{-4}(-2i\log(\sqrt{v+1}+\sqrt{v}))^{-1} \\
&\quad \cdot \int_{-\infty}^{\infty}(h'(\tau+2i)(\tau+2i)+h(\tau+2i))(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau \\
&= (\sqrt{v+1}+\sqrt{v})^{-4}(-2i\log(\sqrt{v+1}+\sqrt{v}))^{-j} \\
&\quad \cdot \int_{-\infty}^{\infty}(h^{(j)}(\tau+2i)(\tau+2i)+jh^{(j-1)}(\tau+2i))(\sqrt{v+1}+\sqrt{v})^{2i\tau}d\tau \\
&\ll_{A,j} (\sqrt{v+1}+\sqrt{v})^{-4}\left(\log(\sqrt{v+1}+\sqrt{v})T\right)^{-j}T.
\end{aligned}$$

In particular, we have a saving in T if $\log(\sqrt{v+1}+\sqrt{v}) \gg T^{-1+\epsilon/2}$. Since $\log(\sqrt{v+1}+\sqrt{v}) = \sqrt{v}+O(v^{3/2})$ for small v, this happens if v or u is $\gg T^{-2+\epsilon}$. We obtain

$$q(v) \ll_{A,j} (\sqrt{v+1}+\sqrt{v})^{-4}T^{1-j\epsilon}.$$

Then

$$\kappa(u) \ll_{A,j} T^{1-j\epsilon}\int_u^{\infty}\frac{dv}{\sqrt{v(v+1)(v-u)}(\sqrt{v+1}+\sqrt{v})^4}.$$

We split the integral in the intervals $]u,u+1[$ and $[u+1,\infty[$. We get

$$\begin{aligned}
\int_u^{\infty}\frac{dv}{\sqrt{v(v+1)(v-u)}(\sqrt{v+1}+\sqrt{v})^4} \ll& \frac{1}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}\int_u^{u+1}\frac{dv}{\sqrt{v-u}}+\int_{u+1}^{\infty}\frac{dv}{v^3} \\
=& \frac{2}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}+\frac{1}{2(u+1)^2}.
\end{aligned}$$

If $u \gg 1$, then we obtain

$$\frac{2}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}+\frac{1}{2(u+1)^2} \ll \frac{1}{u^2}.$$

If $u \ll 1$, then we have $1+u \asymp 1$ and

$$\frac{2}{\sqrt{u(u+1)}(\sqrt{u+1}+\sqrt{u})^4}+\frac{1}{2(u+1)^2} \ll \frac{1}{\sqrt{u}}+1.$$

In summary, for $u \gg T^{-2+\epsilon}$, we computed

$$\begin{aligned}
\kappa(u) &\ll_{A,j} T^{1-j\epsilon}\frac{1}{u^2} && \text{if } u \gg 1, \\
\kappa(u) &\ll_{A,j} T^{2-j\epsilon} && \text{if } u \ll 1.
\end{aligned}$$

Applying Lemma 0.2 below, we sum over γ. For $T^{-2+\epsilon} \ll u_T \ll 1$, we have

$$\sum_{\substack{\gamma\in\mathrm{SL}_2(\mathbb{Z}) \\ T^{-2+\epsilon}\ll u(z_T,\gamma z_T)\ll 1}} |\kappa(u(z_T,\gamma z_T))| \ll_{A,j} T^{2-j\epsilon}(\mathrm{Im}(z)+1)$$

So for j big enough, we can cancel all the powers of k. For $u \gg 1$, we split into dyadic intervals. We can begin the sum at say 1. For $X \gg 1$, we have $\sqrt{X(X+1)}\,\mathrm{Im}(z)+X+1 \ll X(\mathrm{Im}(z)+1)$. We get

$$\begin{aligned}\sum_{\substack{\gamma\in \mathrm{SL}_2(\mathbb{Z})\\ u(z_T,\gamma z_T)\geq 1}} |\kappa(u(z_T,\gamma z_T))| &= \sum_{n=0}^{\infty} \sum_{u\in[2^n,2^{n+1}[} \kappa(u)\\ &\ll \sum_{n=0}^{\infty} 2^n(\mathrm{Im}(z)+1)\kappa(2^n)\\ &\ll_{A,j} \sum_{n=0}^{\infty} T^{1-j\epsilon}(\mathrm{Im}(z)+1)2^{-n}\\ &\ll_{A,j} T^{1-j\epsilon}(\mathrm{Im}(z)+1).\end{aligned}$$

We take j big enough to conclude the proof. □

Remark. We can improve the conditions on $h(\tau)$. If we ask for it to be holomorphic in the strip $\mathrm{Im}(\tau)\leq 1+\delta$, $\delta>0$, the proof works again but the bound blows up as $\delta\to 0$. Conversely, if we have a larger strip, we gain nothing in the bound with this proof.

Lemma 0.2 ([Iwa02], Lemma 2.11)**.** *Let $z\in\mathbb{H}$ with $\mathrm{Im}(z)\geq 1/10$ and $X>0$. We have*

$$\begin{aligned}\#\{\gamma\in \mathrm{SL}_2(\mathbb{Z}) \mid u(z,\gamma z)<X\} &\ll \sqrt{X(X+1)}\,\mathrm{Im}(z)+X+1,\\ \#\{\gamma\in \mathrm{SL}_2(\mathbb{Z}) \mid u(z,\gamma(-\bar{z}))<X\} &\ll \sqrt{X(X+1)}\,\mathrm{Im}(z)+X+1.\end{aligned}$$

References

[Fel23] Gilles Felber. A restriction norm problem for Siegel modular forms. August 2023. arXiv:2308.13493.

[Iwa02] Henryk Iwaniec. *Spectral Methods of Automorphic Forms.* American Mathematical Society, November 2002.