HOMEWORK 1 - DUE ON 06/16/2025

(1) (a) Find $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{30}$ in Cartesian coordinates.
(b) Solve $z^4 + 1 = 0$ in $\mathbb{C}$.

(2) Let n be a positive integer and w a non-zero complex number. Show that there are exactly n complex numbers z such that $z^n = w$. Describe geometrically the numbers z for $w = 1$. In particular, the n-th root function $\sqrt[n]{\cdot}$ is only defined for (positive) real numbers. *Hint: use polar coordinates.*

(3) Let n be a positive number and $w_0 = 1, w_1, \dots, w_{n-1}$ be the n-th root of unity (i.e. $w_k^n = 1$). Show that for any integer m, we have

$$\sum_{k=0}^{n-1} w_k^m = \begin{cases} n & \text{if } n \mid m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

Hint: use that the map $w_k \to e^{2\pi i/n} w_k$ *permutes the roots.*

(4) Let $M \subseteq \mathbb{C}$ be an arbitrary subset of the complex plane. Show the following:
(a) $\partial M = \overline{M} \cap \overline{M^c}$.
(b) $\partial \overline{M} \subseteq \partial M$.
For part a), use the disk characterization of the closure. For part b), express boundary from closure and interior.

(5) Describe in geometric terms and draw a picture of the set of complex numbers z satisfying the following equations.
(a) $1 < |z - i| < 2$,
(b) $|z - 1| = |z - i|$,
(c) $\bar{z} = \dfrac{4}{z}$,
(d) $\operatorname{Im}\left(\frac{z-2}{3}\right) > 0$.

(6) Let $M \subseteq \mathbb{C}$ be an arbitrary set. Show that the following are equivalent:
(a) M is open and closed.
(b) $\partial M = \emptyset$.
(c) $M = \emptyset$ or $M = \mathbb{C}$.