HOMEWORK 2 - DUE ON 06/20/2025

(1) Recall that a region is a complex set that is open and connected. True or false? If true, prove it. If false, give a counterexample.
  (a) The intersection of two regions is a region.
  (b) If two regions intersects, then their union is a region.

(2) (a) Let $f, g : \mathbb{C} \to \mathbb{C}$. Prove that $f \sim g$ as $z \to z_0$ if and and only if $\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1$.
  (b) Let $f, g : \mathbb{R} \to \mathbb{R}_{>0}$ be two functions such that $f \sim g$ as $x \to \infty$ and such that $f(x), g(x) \to \infty$ as $x \to \infty$. Prove that $\log(f) \sim \log(g)$ as $x \to \infty$.
  (c) If $f \sim g$ as $z \to z_0$, is it true that $e^f \sim e^g$?
  (d) Let $f, g : \mathbb{C} \to \mathbb{C}$ holomorphic. Show that $(fg)' = f'g + fg'$ using the following definition of derivative:

$$f(z+h) = f(z) + hf'(z) + o(h) \quad \text{as } h \to 0.$$

(3) Let $(x_n), (y_n) \subseteq \mathbb{R}$ be two sequences of real numbers.
  (a) Show that $\limsup_{n\to\infty}(x_n + y_n) \leq (\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n)$. Is there a similar formula for $\liminf_{n\to\infty}(x_n + y_n)$?
  (b) Give an example where the inequality in (a) is strict.

(4) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ amd $g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$ be two power series with positive radius of convergence. Assume that $f(z_k) = g(z_k)$ for a sequence of complex numbers $z_k \neq a$ that converges to a. Prove that $a_n = b_n$ for all n. *Hint: show that $a_0 = b_0$, then use induction on n.*

(5) Determine the domain and range of the following complex functions:
  (a) $f(z) = e^z$.
  (b) $f(z) = \dfrac{\sin(z)}{\cos(z)}$.
Hint: for (a), use Cartesian coordinates for z and determines the polar coordinates of $f(z)$. For (b), express $f(z)$ in terms of the exponential function and use (a).

(6) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series with positive radius of convergence R. Show that $f(z)$ has an antiderivative in $D(a, R)$, i.e. there exists a holomorphic function $g : D(a, R) \to \mathbb{C}$ such that $g'(z) = f(z)$. *Hint: the antiderivative is another power series.*

(7) For each of the following series, the radius of convergence is $R = 1$. However, they behave differently for $|z| = R = 1$. Show that:
  (a) The power series $\sum nz^n$ does not converge on any point of the unit circle.
  (b) The power series $\sum \frac{z^n}{n^2}$ converges at any point of the unit circle.
  (c) We know from calculus that the power series $\sum \frac{z^n}{n}$ converges at $z = -1$ and diverges at $z = 1$. What happens for $z = i$?

(8) Consider the series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Use the multiplication of power series to show that $e^w \cdot e^z = e^{w+z}$. *Hint: group terms by total power in w and z.*