HOMEWORK 3 - DUE ON 06/27/2025

(1) Compute the length of the following curves.
  (a) The segment $[w, z]$ for $w, z \in \mathbb{C}$.
  (b) The circle of center $z \in \mathbb{C}$ and radius $r > 0$.
  (c) The curve $u : [0, 44] \to \mathbb{C}$ given by $u(t) = t + it^{3/2}$.

(2) Let $f(z_0 + h) = o(1)$ as $h \to 0$. Show that
$$\int_{[z_0, z_0+h]} f = o(h)$$
for h small enough.

(3) Compute
$$\int_{|z|=1} \left(\frac{1}{z} + e^z\right) dz \text{ and } \int_{|z-2|=1} \left(\frac{1}{z} + e^z\right) dz.$$
Hint: the only computation of integral that you need was done in class.

(4) Let γ be the positively oriented circle $|z - 1| = 1$. Show that
$$\int_\gamma \frac{dz}{z^2 - 1} = i\pi.$$
Hint: decompose the integrand into partial fractions.

(5) Let γ be the positively oriented circle $|z| = 1$. Compute
$$\int_\gamma \frac{e^z}{z^4} dz.$$
Hint: use the power series of e^z and split between a part that is holomorphic on $\mathbb{C}$ and the rest.

(6) Show that
$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \sqrt{\frac{\pi}{8}}.$$
Hint: consider the integral of e^{iz^2} on the contour given by the segment $[0, R]$, the circle arc from R to $Re^{i\pi/4}$ and the segment $[Re^{i\pi/4}, 0]$ and let $R \to \infty$.

(7) Let $M \subseteq \mathbb{C}$ be a simply connected region and $f : M \to \mathbb{C}\backslash\{0\}$. Show that for any integer $n \geq 1$ there are exactly n functions $g : M \to \mathbb{C}\backslash\{0\}$ such that $g^n = f$. *Hint: think of $f(z)$ as $e^{h(z)}$ and $g(z)$ as $e^{j(z)}$.*

(8) Let γ be the positively oriented circle $|z| = 1$ and $a, b \in \mathbb{C}$ with $|a| < 1 < |b|$. Show that
$$\int_\gamma \frac{dz}{(z - a)(z - b)} = \frac{2\pi i}{a - b}.$$
Hint: apply Cauchy's formula.