HOMEWORK 4

(1) Let $w \in D(0,1)$. Consider the function

$$g_w(z) := \frac{w-z}{1-\bar{w}z}$$

on $D(0,1)$. Show that g_w is a holomorphic bijection from $D(0,1)$ to $D(0,1)$. *Hint: verify that g_w maps $D(0,1)$ to itself by proving $(w-z)(\bar{w}-\bar{z}) < (1-\bar{w}z)(1-z\bar{w})$. Verify that $g_w \circ g_w(z) = z$ for any $z \in D(0,1)$.*

(2) Let $f : D(a,r) \to \mathbb{C}$ be a holomorphic function. Suppose that f has no holomorphic extension to any disk $D(a,R)$ with $R > r$. Show that r is the radius of convergence of the Taylor series of f at a. *Hint: the problem has a quick solution if you quote the right theorems from the class. Show that the radius of convergence is at least r. Then show it can't be bigger than r.*

(3) Let $f : D(a,r) \to \mathbb{C}$ be a holomorphic function. show that f has a holomorphic extension to $\mathbb{C}$ if and only if $\sqrt[n]{\left|f^{(n)}(a)\right|} = o(n)$ as $n \to \infty$. *Hint: combine the theorem on holomorphicity of power series with the Cauchy-Hadamard formula.*

(4) Prove the Cauchy inequalities: for a holomorphic function $f : M \to \mathbb{C}$ on a region M and $\bar{D}(z_0,R) \subseteq M$, we have

$$\left|f^{(n)}(z_0)\right| \le \frac{n!\,\|f\|_{\partial D(z_0,R)}}{R^n}$$

where $\|f\|_{\partial D(z_0,R)} = \sup_{z\in\partial D(z_0,R)} |f(z)|$ denote the supremum norm of f on $\partial D(z_0,R)$.

(5) Let $f : D(a,R) \to \mathbb{C}$ be a holomorphic function. Show that the range of f has diameter at least $2R\,|f'(a)|$. *Hint: consider $g(z) = f(a+z) - f(a-z)$ and estimate $g'(0)$ via the Cauchy inequalities.*

(6) Let $n > 0$ be an integer and $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)/z^n| \to 0$ as $|z| \to \infty$. Show that z is a polynomial of degree less than n. *Hint: mimic the proof of Liouville's theorem, i.e. bound the n-th coefficient of the Taylor series using Cauchy's formula.*

(7) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function mapping $\mathbb{R}$ to $\mathbb{R}$. Show that if z is a zero of f, then so is $\bar{z}$. *Hint: apply the identity theorem (Chapter 2, Corollary 4.9 in the book) to $f(z)$ and $\overline{f(\bar{z})}$.*

(8) Is there a holomorphic function on a region containing the origin such that:
   (a) $f(\frac{1}{n}) = \frac{1}{2n+1}$?
   (b) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{2n+1}$?