HOMEWORK 5 - DUE ON 07/18/2025

(1) Let f be an entire function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Show that f is constant.

(2) Let $f : M \to \mathbb{C}$ be a function that is holomorphic apart from singularities in M that are all poles. Show that the number of singularities inside a compact set is finite.

(3) Find the singularities of $\frac{1}{\sin(z)}$. Show that they are poles. Give the residue and the order of each pole.

(4) Find the singularities and the principal parts of the following function:
   (a) $(1+z^4)^{-1}$,
   (b) $\frac{1-\cos(z)}{z^2}$,
   (c) $\frac{\sin(z)^2}{z^5}$,
   (d) $(1+z^2)^{-2}$.

(5) Let f be a holomorphic function with a simple pole at z_0 and g be a holomorphic function in a neighborhood of z_0. Show that

$$\operatorname{res}_{z_0}(fg) = g(z_0)\operatorname{res}_{z_0} f.$$

(6) Show that the Laurent series expansion is unique. That is, show that if

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$$

on some annulus centered at z_0, then $a_n = b_n$ for all n. *Hint: consider a contour integral around z_0 and use that the Laurent series converge uniformly in the annulus.*

(7) Consider the following integrals from homework 3. Compute them using the residue theorem:

$$\int_{|z|=1} \left(\frac{1}{z} + e^z\right) dz, \qquad \int_{|z-1|=1} \frac{dz}{z^2-1}, \qquad \int_{|z|=1} \frac{e^z}{z^4} dz.$$

(8) Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

by integrating the function $f(z) = \frac{1}{z^2+1}$ on the contour γ consisting of the real segment $[-R, R]$ and the upper semicircle going from R to $-R$ with center 0.