HOMEWORK 6 - DUE ON 07/24/2025

(1) Find the Laurent series of $\frac{1}{(z-1)(z-2)}$ in the annuli $A(0,0,1)$, $A(0,1,2)$ and $A(0,2,\infty)$. *Hint: use partial fractions and write each term as a geometric series.*

(2) Use any method you like to find ($n > 0$ is an integer)

(a)

$$\int_{|z+1|=2} \frac{e^z}{(z+1)^{34}} dz,$$

(b)

$$\int_{|z-1|=1} \left(\frac{z}{z-1}\right)^n dz.$$

(3) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that $f(1/z)$ has a pole at $z = 0$ if and only if f is a non-constant polynomial. *Hint: start from the Taylor series of $f(z)$ at the origin.*

(4) Prove the identity

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}$$

for $z \in \mathbb{C}\backslash\mathbb{Z}$. *Bonus:* deduce that $\zeta(2) = \frac{\pi^2}{6}$. *Hint: show that the difference of the two sides extends to a bounded entire function whose limit as $\operatorname{Im}(z) \to \pm\infty$ is 0.*

(5) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

Hint: follow the method of exercise (8) in homework 5.

(6) Construct a function f that has a non-isolated singularity and countably many singularities. *Hint: consider the z such that $e^{1/z}$ is 1.*

(7) Let $f(z) = \frac{(z-2)^3 e^z}{(z-1)^4}$. Find

$$\int_{|z|=r} \frac{f'(z)}{f(z)} dz$$

for $r = 3$ and $r = 3/2$.

(8) Let $f(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial. Show that if $|a_k|\, r^k > \sum_{j \neq k} |a_j|\, r^j$ for some $r > 0$, then f has exactly k roots inside $D(0,r)$. *Hint: use Rouché's theorem.*