HOMEWORK 7 - BONUS

(1) Find the maximum of $z^2 - 3iz$ on $\bar{D}(0,1)$.

(2) Use the Cauchy inequalities or the maximum modulus principle to show the following. Let $A, B > 0$ two constants and k an integer. If f is an entire function such that

$$\max_{|z|=R} |f(z)| \leq AR^k + B$$

for all $R > 0$, then f is a polynomial of degree at most k.

(3) Consider the function

$$f(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}.$$

Show that f converges in the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and that it is a holomorphic function there. *Remark: such a function is called a theta series. It is an example of modular form.*

(4) Consider the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n - 1}.$$

(a) Show that f is holomorphic on the disk $D(0,2)$.
(b) Show that f extends uniquely to a meromorphic function F on $\mathbb{C}$. *Hint: compute* $f(2z) - f(z)$.
(c) Find all poles and residues of F.

(5) Let f be a holomorphic function with a Taylor series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Suppose that $a_0 = a_1 = \cdots = a_N = 0$. Show that there is a function g such that $g^N = f$. How many distinct possibilities for g are there?

(6) Find a biholomorphic function between the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and the slit plane $\mathbb{C} \backslash (-\infty, 0]$.

(7) Let M be a bounded region. Let I be an open segment in M that cuts M into two regions M_1 and M_2 (i.e. $M \backslash I = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$). Show that if M_1 and M_2 are simply connected, then M is simply connected.

(8) Weierstrass' theorem states that a continuous function on $[0,1]$ can be uniformly approximated by polynomials. Can every function on the closed disk $D(0,1)$ be uniformly approximated by complex polynomials? Or by general holomorphic function on $D(0,1)$?