SOLUTION HOMEWORK 1

(1) (a) Find $(\frac{1}{2} + \frac{\sqrt{3}}{2} i)^{30}$ in Cartesian coordinates.
(b) Solve $z^4 + 1 = 0$ in $\mathbb{C}$.

Solution:
(a) We have $z = \frac{1}{2} + \frac{\sqrt{3}}{2} i = e^{i\pi/3}$. Then $z^{30} = e^{10i\pi} = 1$.
(b) Write $z = re^{it}$. Then $-1 = e^{i\pi} = r^4 e^{4it}$. We deduce that $r = 1$, since $r > 0$, and $4t = \pi \pmod{2\pi}$. For $0 \leq t < 2\pi$, this gives the solutions $t = \frac{\pi}{4} + k\frac{\pi}{2}$, $k = 0, 1, 2, 3$.

(2) Let n be a positive integer and w a non-zero complex number. Show that there are exactly n complex numbers z such that $z^n = w$. Describe geometrically the numbers z for $w = 1$. In particular, the n-th root function $\sqrt[n]{\cdot}$ is only defined for (positive) real numbers. *Hint: use polar coordinates.*

Solution: Suppose that $w = re^{it}$. Then the numbers $z_n = r^{1/n} e^{i(\frac{t}{n} + \frac{2\pi k}{n})}$ are clearly distinct solutions for $k = 0, 1, \ldots, n-1$. Here $r^{1/n}$ is the positive n-th root of r, well defined since it is a real positive number. Moreover, a degree n polynomial can't have more than n roots so they are the only ones. More precisely, we can write

$$z^n - w = \prod_{k=0}^{n-1} \left(z^n - r^{1/n} e^{i(\frac{t}{n} + \frac{2\pi k}{n})} \right).$$

If z is such that $z^n - w = 0$, then it must cancel one of the factors on the right and so be one of the roots given above.

(3) Let n be a positive number and $w_0 = 1, w_1, \ldots, w_{n-1}$ be the n-th root of unity (i.e. $w_k^n = 1$). Show that for any integer m, we have

$$\sum_{k=0}^{n-1} w_k^m = \begin{cases} n & \text{if } n \mid m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

Hint: use that the map $w_k \to e^{2\pi i/n} w_k$ permutes the roots.

Solution: Clearly, if $n \mid m$, then $w_k^m = 1$ for all k and the sum is n. Note that $w_1 \cdot w_k = w_{k+1}$ and $w_1 \cdot w_{n-1} = 1$. So the multiplication by w_1 permutes all the roots. If $n \nmid m$, then $w_1^m \neq 1$. We compute

$$w_1^m \sum_{k=0}^{n-1} w_k^m = \sum_{k=0}^{n-1} (w_k w_1)^m = \sum_{k=0}^{n-1} w_k^m.$$

So the sum must be 0.

(4) Let $M \subseteq \mathbb{C}$ be an arbitrary subset of the complex plane. Show the following:
(a) $\partial M = \overline{M} \cap \overline{M^c}$.
(b) $\partial \overline{M} \subseteq \partial M$.
For part a), use the disk characterization of the closure. For part b), express boundary from closure and interior.

Solution:

(a) Note that $z \notin \text{int}(M) \Leftrightarrow \forall r > 0\ D(z,r) \not\subseteq M \Leftrightarrow \forall r > 0\ D(z,r) \cap M^c \neq \emptyset$. Then

$$\begin{aligned} z \in \overline{M} \cap \overline{M^c} &\Leftrightarrow z \in \overline{M} \text{ and } z \in \overline{M^c} \\ &\Leftrightarrow z \in \overline{M} \text{ and } \forall r > 0\ D(z,r) \cap M^c \neq \emptyset \\ &\Leftrightarrow z \in \overline{M} \text{ and } z \notin \text{int}(M) \\ &\Leftrightarrow z \in \overline{M} \backslash \text{int}(M). \end{aligned}$$

Note that in particular, we showed that $\text{int}(M)^c = \overline{M^c}$.

(b) Claim: we have $\text{int}(M) \subseteq \text{int}(\overline{M})$. This is because if $z \in \text{int}\, M$, there exists $r > 0$ with $D(z,r) \subseteq M \subseteq \overline{M}$. Moreover $\overline{\overline{M}} = \overline{M}$, since $\overline{M}$ is closed. Therefore

$$\partial \overline{M} = \overline{\overline{M}} \backslash \text{int}(\overline{M}) \subseteq \overline{M} \backslash \text{int}(M) = \partial M.$$

(5) Describe in geometric terms and draw a picture of the set of complex numbers z satisfying the following equations.

(a) $1 < |z - i| < 2$,

(b) $|z - 1| = |z - i|$,

(c) $\bar{z} = \dfrac{4}{z}$,

(d) $\text{Im}\left(\frac{z-2}{3}\right) > 0$.

Solution:

(a) Open annulus of center i and of radii 1 and 2.

(b) Line $x = y$. These are the points at the same distance from 1 and i.

(c) Circle centered at 0 and of radius 2.

(d) Open upper half-plan $\text{Im}(z) > 2$.

(6) Let $M \subseteq \mathbb{C}$ be an arbitrary set. Show that the following are equivalent:

(a) M is open and closed.

(b) $\partial M = \emptyset$.

(c) $M = \emptyset$ or $M = \mathbb{C}$.

Solution:

$(a) \Leftrightarrow (b)$: M is open and closed $\Leftrightarrow \text{int}(M) = \overline{M} \Leftrightarrow \partial M = \overline{M} \backslash \text{int}(M) = \emptyset$.

$(a) \Rightarrow (c)$: note that $\mathbb{C}$ is path-connected: given $z_1, z_2 \in \mathbb{C}$, the segment $[z_1, z_2]$, parametrized by $u(t) = (1-t)z_1 + tz_2$, $t \in [0,1]$, connects the two points. In particular, $\mathbb{C}$ is connected. If $M \neq \mathbb{C}, \emptyset$ and M is open and closed, then $M^c \neq \emptyset$ and M^c is open. Then $\mathbb{C} = M \cup M^c$ and $M \cap M^c = \emptyset$, contradicting the connectedness of $\mathbb{C}$.

Other solution : by contradiction, suppose that M is open and closed and that there exists $z_1 \in M$ and $z_2 \notin M$. Consider the segment $[z_1, z_2]$ parametrized by $u(t) = (1-t)z_1 + tz_2$, $t \in [0,1]$. Consider $t_0 = \sup\{t \in [0,1] : u(t) \in M\}$. If $t_0 = 0$ resp. 1, then $z_1 \in \partial M$ resp. $z_2 \in \partial M$. Contradiction. Suppose that $t_0 \in (0,1)$. First suppose that $u(t_0) \in M$. Since M is open, there exists $r > 0$ such that $D(u(t_0), r) \subseteq M$. Consider $\delta = \frac{r}{2|z_2 - z_1|}$ and $t_1 = t_0 + \delta$. Claim: $u(t_1) \in M$. We compute

$$|u(t_1) - u_t(_0)| = |(1 - t_0 - \delta)z_1 + (t_0 + \delta)z_2 - (1 - t_0)z_1 - t_0 z_2| = \delta\, |z_1 - z_2| = \frac{r}{2}.$$

So $u(t_0 + \delta) \in M$. Contradiction with the definition of t_0. If $u(t_0) \notin M$, note that M^c is open since M is closed. The same reasoning with a disk in M^c leads to a contradiction.

$(c) \Rightarrow (a)$: both set are clearly open. Moreover they are complement of each other so they are also closed.