SOLUTION HOMEWORK 2

(1) Recall that a region is a complex set that is open and connected. True or false? If true, prove it. If false, give a counterexample.
  (a) The intersection of two regions is a region.
  (b) If two regions intersects, then their union is a region.

Solution: In both cases, it is clear that the resulting set is open.
  (a) False: consider the annulus sectors
$$M_1 = \{re^{it} \mid r \in (1,2),\ t \in (0, 3\pi/2)\}, \quad M_2 = \{re^{it} \mid r \in (1,2),\ t \in (-\pi, \pi/2)\}.$$
  (b) True: we show that the resulting set is path-connected assuming that the two original sets M_1, M_2 are so. Let $w, z \in M_1 \cup M_2$. Then if w and z are both in M_1 or in M_2, there is a path from w to z since M_1 resp. M_2 is path-connected. Otherwise, WLOG assume that $w \in M_1$ and $z \in M_2$. Let $w' \in M_1 \cap M_2$ which is not empty by hypothesis. Then $w' \in M_1$ resp. M_2 implies that there is a path from w to w' resp. from w' to z. Joining these two paths gives a path from w to z.

(2) (a) Let $f, g : \mathbb{C} \to \mathbb{C}$. Prove that $f \sim g$ as $z \to z_0$ if and and only if $\lim_{z\to z_0} \frac{f(z)}{g(z)} = 1$.
  (b) Let $f, g : \mathbb{R} \to \mathbb{R}_{>0}$ be two functions such that $f \sim g$ as $x \to \infty$ and such that $f(x), g(x) \to \infty$ as $x \to \infty$. Prove that $\log(f) \sim \log(g)$ as $x \to \infty$.
  (c) If $f \sim g$ as $z \to z_0$, is it true that $e^f \sim e^g$?
  (d) Let $f, g : \mathbb{C} \to \mathbb{C}$ holomorphic. Show that $(f+g)' = f'g + fg'$ using the following definition of derivative:
$$f(z+h) = f(z) + hf'(z) + o(h) \quad \text{as } h \to 0.$$

Solution:
  (a) We have
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = 1 + \lim_{z\to z_0} \frac{f(z) - g(z)}{g(z)}.$$
The limit is equal to 1 if and only if $f - g = o(g)$.
  (b) We have
$$\log(f(x)) - \log(g(x)) = \log\left(\frac{f(x)}{g(x)}\right) = \log\left(1 + \frac{f(x) - g(x)}{g(x)}\right)$$
Since $\log(1+x) = O(x)$, we have
$$\log(f(x)) - \log(g(x)) = O\left(\frac{f(x) - g(x)}{g(x)}\right) = o(1).$$
Since $g(x) \to \infty$, this is $o(g)$.
  (c) False: consider $f(z) = z^2$ and $g(z) = z^2 + z$ as $z \to \infty$. Then $f(z) - g(z) = z = o(g(z))$ but $e^{g(z)} = e^z f(z)$ and so $\lim_{z\to\infty} \frac{f(z)}{g(z)} = 0$.

(d) We compute

$$\begin{aligned} f(z+h)g(z+h) &= (f(z)+hf'(z)+o(h))(g(z)+hg'(z)+o(h)) \\ &= f(z)g(z)+h(f'(z)g(z)+f(z)g'(z)) \\ &\quad + h^2 f'(z)g'(z) + o(h)(f(z)+hf'(z)+g(z)+hg'(z)+o(h)). \end{aligned}$$

Since z is fixed and $h \to 0$, everything on the second line is $o(h)$. We get

$$f(z+h)g(z+h) = f(z)g(z) + h(f'(z)g(z)+f(z)g'(z)) + o(h).$$

That is $(fg)' = f'g + fg'$.

(3) Let $(x_n), (y_n) \subseteq \mathbb{R}$ be two sequences of real numbers.
  (a) Show that $\limsup_{n\to\infty}(x_n+y_n) \le (\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n)$. Is there a similar formula for $\liminf_{n\to\infty}(x_n+y_n)$?
  (b) Give an example where the inequality in (a) is strict.

Solution:
(a) By definition and subadditivity of sup, we have

$$\begin{aligned} \limsup_{n\to\infty}(x_n+y_n) &= \lim_{m\to\infty} \sup_{n\ge m}(x_n+y_n) \\ &\le \lim_{m\to\infty} (\sup_{n\ge m} x_n + \sup_{n\ge m} y_n) \\ &\le \lim_{m\to\infty} \sup_{n\ge m} x_n + \lim_{m\to\infty} \sup_{n\ge m} y_n. \end{aligned}$$

Yes, $\liminf_{n\to\infty}(x_n+y_n) \ge (\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n)$. The proof is the same as for lim sup using superadditivity of inf.
(b) For example $x_n = (-1)^n$, $y_n = -(-1)^n$. Then $x_n + y_n = 0$ for all n but $\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = 1$.

(4) Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-a)^n$ be two power series with positive radius of convergence. Assume that $f(z_k) = g(z_k)$ for a sequence of complex numbers $z_k \neq a$ that converges to a. Prove that $a_n = b_n$ for all n. *Hint: show that $a_0 = b_0$, then use induction on n.*

Solution: Since f and g have both a positive radius of convergence, they are continuous. In particular, $f(a) = g(a)$. Therefore $a_0 = b_0$. Then $f(z) - a_0 = (z-a)f_1(z)$ for the function $f_1(z) = \sum_{n=0}^{\infty} a_{n+1}(z-a)^n$ with the same radius of convergence. Define $g_1(z)$ in the same fashion. Clearly

$$f_1(z_k) = \frac{f(z_k)-a_0}{(z_k-a)} = \frac{g(z_k)-b_0}{(z_k-a)} = g_1(z_k).$$

So f_1 and g_1 satisfy the same conditions as f and g. Therefore $a_1 = b_1$. By induction, we see that $a_n = b_n$ for all n.

(5) Determine the domain and range of the following complex functions:
  (a) $f(z) = e^z$.
  (b) $f(z) = \dfrac{\sin(z)}{\cos(z)}$.

Hint: for (a), use Cartesian coordinates for z and determines the polar coordinates of $f(z)$. For (b), express $f(z)$ in terms of the exponential function and use (a).

Solution:

(a) The domain of f is $\mathbb{C}$. Let $z = x + iy$. Then $e^z = e^x e^{iy}$. Given $w = re^{it} \in \mathbb{C}$ with $r \neq 0$, consider $x = \log(r)$ and $y = t$. We see that $e^z = w$. Moreover, if $e^z = 0$, then $e^{\mathrm{Re}(z)} = 0$ which is not possible, by classical analysis. So e^z has range $\mathbb{C}\backslash\{0\}$.

(b) We have

$$if(z) = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{e^{2iz} - 1}{e^{2iz} + 1} = 1 - \frac{2}{e^{2iz} + 1}.$$

We see that the domain of f is given by the z such that $e^{2iz} \neq -1$, that is $z \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. The function $e^{2iz} + 1$ has range $\mathbb{C}\backslash\{1\}$ by the last exercise. So $\frac{2}{e^{2iz}+1}$ has range $\mathbb{C}\backslash\{0, 2\}$ and f has range $\mathbb{C}\backslash\{-i, i\}$.

(6) Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series with positive radius of convergence R. Show that $f(z)$ has an antiderivative in $D(a, R)$, i.e. there exists a holomorphic function $g : D(a, R) \to \mathbb{C}$ such that $g'(z) = f(z)$. *Hint: the antiderivative is another power series.*

Solution: Consider the power series

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}.$$

Differentiating $g(z)$ termwise, we obtain $f(z)$. By a theorem seen in class, the radius of convergence of g and f are the same, g is holomorphic and $g'(z) = f(z)$.

(7) For each of the following series, the radius of convergence is $R = 1$. However, they behave differently for $|z| = R = 1$. Show that:

(a) The power series $\sum nz^n$ does not converge on any point of the unit circle.
(b) The power series $\sum \frac{z^n}{n^2}$ converges at any point of the unit circle.
(c) We know from calculus that the power series $\sum \frac{z^n}{n}$ converges at $z = -1$ and diverges at $z = 1$. What happens for $z = i$?

Solution:

(a) For $|z| = 1$, the term nz has absolute value going to infinity, so the series isn't a Cauchy sequence and can't converge.
(b) If $|z| = 1$, then $\left|\sum \frac{z^n}{n^2}\right| \leq \sum \frac{1}{n^2} = \frac{\pi^2}{6}$. In other words, the series converges absolutely.
(c) Splitting between real and imaginary parts, that is between even and odd indices, we have

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Both terms are alternating series with decreasing coefficients. By a theorem of analysis, they both converge. More precisely, we have

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = -\frac{\log(2)}{2} + i\frac{\pi}{4}.$$

(8) Consider the series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Use the multiplication of power series to show that $e^w \cdot e^z = e^{w+z}$. *Hint: group terms by total power in w and z.*

Solution: We compute

$$\begin{aligned}\left(\sum_{m=0}^{\infty}\frac{w^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right) &= \sum_{k=0}^{\infty}\sum_{m+n=k}\frac{w^m z^n}{m!n!} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{m+n=k}\frac{(m+n)!}{m!n!}w^m z^n \\ &= \sum_{k=0}^{\infty}\frac{(wz)^k}{k!}.\end{aligned}$$

We used the binomial theorem on the last line.