SOLUTION HOMEWORK 3

(1) Compute the length of the following curves.

(a) The segment $[w, z]$ for $w, z \in \mathbb{C}$.

(b) The circle of center $z \in \mathbb{C}$ and radius $r > 0$.

(c) The curve $u : [0, 44] \to \mathbb{C}$ given by $u(t) = t + it^{3/2}$.

Solution:

(a) We parametrize the segment by

$$u : [0, 1] \to \mathbb{C},\ t \mapsto (1 - t)w + tz.$$

We have $u'(t) = -w + z$. Then

$$\ell(u) = \int_0^1 |u'(t)|\, dt = |z - w| \int_0^1 dt = |z - w|\,.$$

(b) We parametrize the circle by

$$u : [0, 2\pi] \to \mathbb{C},\ t \mapsto z + re^{it}.$$

We have $u'(t) = rie^{it}$. Then

$$\ell(u) = \int_0^{2\pi} |u'(t)|\, dt = r \int_0^{2\pi} dt = 2\pi r.$$

(c) We have $u'(t) = 1 + i\frac{3}{2}t^{1/2}$. Then

$$\ell(u) = \int_0^4 4\sqrt{1 + \frac{9t}{4}} dt = \frac{2}{3} \cdot \frac{4}{9} \left(1 + \frac{9}{4}t\right)^{3/2}\Bigg|_0^{44} = \frac{2}{3}\frac{4}{9}(1 + 99)^{3/2} = \frac{8}{27}(1000 - 1) = 296.$$

(2) Let $f(z_0 + h) = o(1)$ as $h \to 0$. Show that

$$\int_{[z_0, z_0+h]} f = o(h)$$

for h small enough.

Solution: Recall that $f(z_0 + h) = o(1)$ as $h \to 0$ means that

$$\forall \epsilon > 0\ \exists \delta > 0 :\ |h| < \delta \Rightarrow |f(z_0 + h)| < \epsilon.$$

Also, $o(h) = h \cdot o(1)$. Let $\epsilon > 0$ be fixed and $\delta > 0$ given by the above formula.

$$\left| \int_{[z_0, z_0+h]} f \right| \le h \sup_{h' \in [0,h]} |f(z_0 + h')|\,.$$

If $|h| < \delta$, then $|h'| < \delta$. So $|f(z_0 + h')| < \epsilon$ and the supremum is smaller or equal to ϵ. We showed that

$$\forall \epsilon > 0\ \exists \delta > 0\ \ |h| < \delta \Rightarrow \left| \int_{[z_0, z_0+h]} f \right| \le h\epsilon,$$

that is the integral is $o(h)$.

(3) Compute

$$\int_{|z|=1} \left(\frac{1}{z} + e^z\right) dz \text{ and } \int_{|z-2|=1} \left(\frac{1}{z} + e^z\right) dz.$$

Hint: the only computation of integral that you need was done in class.

Solution: We saw that

$$\int_{|z|=1} \frac{1}{z} dz = 2\pi i.$$

Moreover, e^z is holomorphic on $\mathbb{C}$. By Cauchy's theorem:

$$\int_{|z|=1} \left(\frac{1}{z} + e^z\right) dz = \int_{|z|=1} \frac{1}{z} dz + \int_{|z|=1} e^z dz = 2\pi i + 0.$$

For the second integral, note that $\frac{1}{z}$ is holomorphic in the interior of the circle $|z-2| = 1$. So by Cauchy's theorem:

$$\int_{|z-2|=1} \left(\frac{1}{z} + e^z\right) dz = 0.$$

(4) Let γ be the positively oriented circle $|z-1| = 1$. Show that

$$\int_\gamma \frac{dz}{z^2-1} = i\pi.$$

Hint: decompose the integrand into partial fractions.

Solution: We have

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

The second part of the integrand is holomorphic in the interior of the circle γ. By Cauchy's theorem and an integral computed in class:

$$\int_\gamma \frac{dz}{z^2-1} = \frac{1}{2}\int_\gamma \frac{dz}{z-1} - \frac{1}{2}\int_\gamma \frac{dz}{z+1} = \pi i + 0.$$

(5) Let γ be the positively oriented circle $|z| = 1$. Compute

$$\int_\gamma \frac{e^z}{z^4} dz.$$

Hint: use the power series of e^z and split between a part that is holomorphic on $\mathbb{C}$ and the rest.

Solution: We have

$$\frac{e^z}{z^4} = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+4)!}.$$

The series on the RHS is holomorphic on $\mathbb{C}$ since

$$\lim_{n\to\infty} ((n+4)!)^{1/n} = \lim_{n\to\infty} (n!)^{1/n} \cdot \lim_{n\to\infty} [(n+1)(n+2)(n+3)(n+4)]^{1/n} = \infty.$$

We compute in class that the integral $\int_\gamma \frac{1}{z^n} dz = 0$ for $n \geq 2$ and it is $2\pi i$ for $n = 1$. By Cauchy's theorem and the integrals computed in class:

$$\int_\gamma \frac{e^z}{z^4} dz = \int_\gamma \left(\frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2z^2}\right) dz + \int_\gamma \frac{1}{6z} dz + \int_\gamma \sum_{n=0}^{\infty} \frac{z^n}{(n+4)!} dz = 0 + \frac{\pi i}{3} + 0.$$

(6) Show that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \sqrt{\frac{\pi}{8}}.$$

Hint: consider the integral of e^{iz^2} on the contour given by the segment $[0,R]$, the circle arc from R to $Re^{i\pi/4}$ and the segment $[Re^{i\pi/4},0]$ and let $R\to\infty$.

Solution: We consider the contour given in the hint. Let γ_R be the circle arc centered at 0 from R to $Re^{i\pi/4}$. Let $f(z)=e^{iz^2}$. Then by Cauchy's theorem:

$$0=\int_{[0,R]} f + \int_{\gamma_R} f + \int_{[Re^{i\pi/4},0]} f.$$

We parametrize γ_R by $u:[0,\pi/4]$, $u(t)=Re^{it}$. We compute

$$\int_{\gamma_R} f = \int_0^{\pi/4} e^{iR^2e^{2it}} Rie^{it}dt.$$

Note that $\mathrm{Im}(e^{2it})=\sin(2t)\geq t$ for $t\in[0,\pi/4]$ (proof: the scond derivative is non-positive on the interval so it is concave). Then

$$\begin{aligned}\left|\int_{\gamma_R} f\right| &\leq R\int_0^{\pi/4}\left|e^{iR^2e^{2it}}\right|dt\\ &\leq R\int_0^{\pi/4} e^{-R^2t}dt\\ &= -R\left.\frac{e^{-R^2t}}{R^2}\right|_0^{\pi/4}\\ &= \frac{1-e^{-\pi R^2/4}}{R}.\end{aligned}$$

This goes to 0 as $R\to\infty$. Therefore we have

$$0=\int_{[0,R]} f + o(1) - \int_{[0,Re^{i\pi/4}]} f.$$

We parametrize the segments as usual:

$$0=\int_0^R e^{ix^2}dx + o(1) - \int_0^R e^{i(xe^{i\pi/4})^2}e^{i\pi/4}dx$$

Taking the limit as $R\to\infty$, we get

$$e^{i\pi/4}\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{ix^2}dx = \int_0^\infty \cos(x^2)dx + i\int_0^\infty \sin(x^2)dx.$$

Recall that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{2}}{2}$. Write $\frac{\sqrt{\pi}}{2}e^{i\pi/4} = \frac{\sqrt{\pi}}{2}\frac{1+i}{\sqrt{2}} = \sqrt{\frac{\pi}{8}}(1+i)$. Taking real and imaginary parts of the above equation, we get

$$\int_0^\infty \cos(x^2)dx = \sqrt{\frac{\pi}{8}},\quad \int_0^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{8}}.$$

(7) Let $M\subseteq\mathbb{C}$ be a simply connected region and $f:M\to\mathbb{C}\backslash\{0\}$. Show that for any integer $n\geq 1$ there are exactly n functions $g:M\to\mathbb{C}\backslash\{0\}$ such that $g^n=f$. *Hint: think of $f(z)$ as $e^{h(z)}$ and $g(z)$ as $e^{j(z)}$.*

Solution: Since f is a non-vanishing function, we saw that there is a holomorphic logarithm $h:M\to\mathbb{C}$ with $e^h=f$. Then $g_k(z)=e^{h/n+2\pi ik/n}$ is such that $g_k^n=f$ for $k=0,\dots,n-1$.

If g is another function such that $g^n = f$, then $g(z) \neq 0$ for all $z \in M$. Therefore, there exists $j : M \to \mathbb{C}$ with $e^j = g$. Then $e^{nj} = f = e^h$. This means that $e^{h-nj} = 1$ is a constant function. Then

$$0 = (e^{h(z)-nj(z)})' = (h'(z) - nj'(z))e^{h(z)-nj(z)} \Rightarrow h'(z) = nj'(z).$$

So $nj(z) = h(z) + c$ for some $c \in \mathbb{C}$. Clearly, $c = 2\pi ik$ for some k. Therefore $g(z) = e^{j(z)+2\pi ik/n}$ is one of the solution g_k given above.

(8) Let γ be the positively oriented circle $|z| = 1$ and $a, b \in \mathbb{C}$ with $|a| < 1 < |b|$. Show that

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}.$$

Hint: apply Cauchy's formula.

Solution: The function $\frac{1}{z-b}$ is holomorphic in the interior of γ. By Cauchy's formula:

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{(z-a)(z-b)} = \left.\frac{1}{z-b}\right|_{z=a} = \frac{1}{a-b}.$$