SOLUTION HOMEWORK 4

(1) Let $w \in D(0,1)$. Consider the function

$$g_w(z) := \frac{w - z}{1 - \bar{w}z}$$

on $D(0,1)$. Show that g_w is a holomorphic bijection from $D(0,1)$ to $D(0,1)$. *Hint: verify that g_w maps $D(0,1)$ to itself by proving $(w-z)(\bar{w}-\bar{z}) < (1-\bar{w}z)(1-z\bar{w})$. Verify that $g_w \circ g_w(z) = z$ for any $z \in D(0,1)$.*

Solution: Note that $0 < (1 - |w|^2)(1 - |z|^2)$ for all $w, z \in D(0,1)$. Expanding, we see that $|w|^2 + |z|^2 < 1 + |wz|^2$. We have

$$(w - z)(\bar{w} - \bar{z}) = |w|^2 + |z|^2 - (\bar{w}z + w\bar{z}) < 1 + |wz|^2 - (\bar{w}z + w\bar{z}) = (1 - \bar{w}z)(1 - z\bar{w}).$$

We deduce that

$$|g_w(z)|^2 = g_w(z)\overline{g_w(z)} = \frac{(w - z)(\bar{w} - \bar{z})}{(1 - \bar{w}z)(1 - w\bar{z})} < 1.$$

So g_w maps $D(0,1)$ to itself. We also have

$$\begin{aligned} g_w \circ g_w(z) &= \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} \\ &= \frac{w(1 - \bar{w}z) - (w - z)}{(1 - \bar{w}z) - \bar{w}(w - z)} \\ &= \frac{z - |w|^2 z}{1 - |w|^2} \\ &= z. \end{aligned}$$

So g_w is a bijection.

(2) Let $f : D(a,r) \to \mathbb{C}$ be a holomorphic function. Suppose that f has no holomorphic extension to any disk $D(a,R)$ with $R > r$. Show that r is the radius of convergence of the Taylor series of f at a. *Hint: the problem has a quick solution if you quote the right theorems from the class. Show that the radius of convergence is at least r. Then show it can't be bigger than r.*

Solution: Let ρ be the radius of convergence of the Taylor series of f. By Chapter 2, Theorem 4.4 in the book, $f(z)$ is equal to its Taylor series on $D(a,r)$ so $\rho \geq r$. By Theorem 2.8 in the notes, the Taylor series defines a holomorphic function on $D(a,\rho)$. Therefore, by the initial assumption on $f(z)$, we can't have $\rho > r$. So $\rho = r$.

(3) Let $f : D(a,r) \to \mathbb{C}$ be a holomorphic function. show that f has a holomorphic extension to $\mathbb{C}$ if and only if $\sqrt[n]{|f^{(n)}(a)|} = o(n)$ as $n \to \infty$. *Hint: combine the theorem on holomorphicity of power series with the Cauchy-Hadamard formula.*

Solution: We know that f can be written as a power series in $D(a,r)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n.$$

Recall that $a_n = \frac{f^{(n)}(a)}{n!}$. Note that (by the last exercise) f can be extended to $\mathbb{C}$ if and only if the radius of convergence R of the Taylor series is infinite. The radius of convergence is given by

$$R^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \frac{\sqrt[n]{\left|f^{(n)}(a)\right|}}{\sqrt[n]{n!}}.$$

The Stirling formula tells us that $\sqrt[n]{|n!|} \sim \frac{n}{e}$ as $n \to \infty$. In particular, $R^{-1} = 0$ if and only if $\frac{\sqrt[n]{\left|f^{(n)}(a)\right|}}{n} \to 0$ as $n \to \infty$, i.e. $\sqrt[n]{\left|f^{(n)}(a)\right|} = o(n)$. Then

$$f \text{ has an extension to } \mathbb{C} \Leftrightarrow R = \infty \Leftrightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0 \Leftrightarrow \sqrt[n]{\left|f^{(n)}(a)\right|} = o(n).$$

(4) Prove the Cauchy inequalities: for a holomorphic function $f : M \to \mathbb{C}$ on a region M and $\bar{D}(z_0, R) \subseteq M$, we habe

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \, \|f\|_{\partial D(z_0,R)}}{R^n}$$

where $\|f\|_{\partial D(z_0,R)} = \sup_{z\in\partial D(z_0,R)} |f(z)|$ denote the supremum norm of f on $\partial D(z_0, R)$.

Solution: By Cauchy's formula, we have

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \frac{n!}{2\pi} \left|\int_{\partial D(z_0,R)} \frac{f(z)}{(z-z_0)^{n+1}} dz\right| \\ &\leq \frac{n!}{2\pi} \int_{\partial D(z_0,R)} \frac{|f(z)|}{|z-z_0|^{n+1}} dz \\ &\leq \frac{n!}{2\pi} \frac{2\pi R}{R^{n+1}} \|f\|_{\partial D(z_0,R)} .\end{aligned}$$

We used that $|z - z_0| = R$ on the contour and the definition of the supremum norm.

(5) Let $f : D(a, R) \to \mathbb{C}$ be a holomorphic function. Show that the range of f has diameter at least $2R\,|f'(a)|$. *Hint: consider $g(z) = f(a + z) - f(a - z)$ and estimate $g'(0)$ via Cauchy's inequalities.*

Solution: Let $g(z) = f(a + z) - f(a - z)$. We have $g'(0) = 2f'(a)$. Consider $0 < r < R$. By Cauchy's inequalities, we have

$$|g'(0)| \leq \frac{\|g\|_{\partial D(a,r)}}{r} = \frac{\sup_{z\in\partial D(a,r)} |f(a+z) - f(a-z)|}{r}.$$

We deduce that

$$\sup_{w,z\in D(z,R)} |f(z) - f(w)| \geq \sup_{z\in\partial D(a,r)} |f(a+z) - f(a-z)| \geq r\,|g'(0)| = 2r\,|f'(a)|$$

for all $0 < r < R$. Since the left-hand side doesn't depend on r, we can take the limit as $r \to R$:

$$\sup_{w,z\in D(z,R)} |f(z) - f(w)| \geq 2R\,|f'(a)|\,.$$

(6) Let $n > 0$ be an integer and $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)/z^n| \to 0$ as $|z| \to \infty$. Show that z is a polynomial of degree less than n. *Hint: mimic the proof of Liouville's theorem, i.e. bound the n-th coefficient of the Taylor series using Cauchy's formula.*

Solution: By hypothesis $|f(z)| = o(z^n)$ as $|z| \to \infty$. We know that f is given by its Taylor series at 0: $f(z) = \sum_{m=0}^{\infty} a_m z^m$ for all $z \in \mathbb{C}$. Using the Cauchy inequalities, we have

$$a_m = \frac{f^{(m)}(0)}{m!} \leq \frac{\|f\|_{\partial D(z_0,R)}}{R^m} = o(R^{n-m})$$

as $R \to \infty$. If $n \leq m$, then $a_m = o(1)$ as $R \to \infty$, so $a_m = 0$. Therefore, only the first $n-1$ coefficients can be non-zero and f is a polynomial of degree less than n.

(7) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function mapping $\mathbb{R}$ to $\mathbb{R}$. Show that if z is a zero of f, then so is $\bar{z}$. *Hint: apply the identity theorem (Chapter 2, Corollary 4.9 in the book) to* $f(z)$ *and* $\overline{f(\bar{z})}$.

We know that f is given by its Taylor series at 0: $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$. Let $g(z) = \overline{f(\bar{z})}$. We compute

$$g(z) = \overline{\sum_{n=0}^{\infty} a_n \bar{z}^n} = \sum_{n=0}^{\infty} \bar{a}_n z^n.$$

By the Cauchy-Hadamard formula, g defines a holomorphic function on $\mathbb{C}$. Moreover $f(z) = g(z)$ for $z \in \mathbb{R}$. In particular, $f(\frac{1}{n}) = g(\frac{1}{n})$. By the identity theorem, we have $f = g$. Now, consider z a zero of f. Then

$$f(\bar{z}) = \overline{g(z)} = \overline{f(z)} = \bar{0} = 0.$$

Note: we also proved that $f(z)$ maps $\mathbb{R}$ to $\mathbb{R}$ if and only if its Taylor coefficients are real.

(8) Is there a holomorphic function on a region containing the origin such that:
(a) $f(\frac{1}{n}) = \frac{1}{2n+1}$.
(b) $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{2n+1}$.

Solution:
(a) Let $z = \frac{1}{n}$, so that $n = \frac{1}{z}$. We want that

$$f(z) = \frac{1}{2\frac{1}{z} + 1} = \frac{z}{z+2}.$$

This function is defined for $z \in \mathbb{C} \setminus \{-2\}$.
(b) Note that $f(-\frac{1}{n}) = -\frac{1}{2n-1} \neq \frac{1}{2n+1}$. Suppose that a function g satisfies the condition of the exercise. Then $f(\frac{1}{n}) = g(\frac{1}{n})$ and 0 is in the domain of f and g. By the identity theorem, we must have $f = g$. So no such function exists.