SOLUTION 5

(1) Let f be an entire function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Show that f is constant.

Solution: By induction, we have $f(z) = f(\tilde{z})$ where $\tilde{z}$ is the unique complex number in $[0,1)^2$ such that $z - \tilde{z}$ has integral real and imaginary parts. Moreover, $[0,1]^2$ is compact so f is bounded by a constant $M > 0$ on it. Hence $|f(\tilde{z})| \leq M$. In conclusion, f is entire and bounded. By Liouville's theorem, it is constant.

(2) Let $f : M \to \mathbb{C}$ be a function that is holomorphic apart from singularities in M that are all poles. Show that the number of singularities inside a compact set is finite.

Solution: Let $K \subseteq M$ be a compact set. Let S be the set of singularities in K. Since they are all isolated poles, for each $z \in S$, there is $r_z > 0$ such that f does not vanish on $\dot{D}(z, r_z)$. Then the set $U = \bigcup_{z \in S} D(z, r_z)$ is open. Thus $K \backslash U = K \cap U^c$ is a closed and bounded set, i.e. it is compact. By the identity theorem, either $f = 0$ or f has finitely many zeros on $K \backslash U$. The first case is trivial. In the second case, we showed that f has finitely many zeros in K. Then $1/f$ is a function with finitely many singularities given by the above zeros, that are poles, in K. In particular, the singularities are isolated. The zeros of $1/f$ are exactly the pole of S. By the same reasoning as above, there are finitely many of them.
Other solution: If f has an infinite amount of pole inside a compact K, then there is a sequence of poles that converges. Then the limit point of these poles is also a pole because any neighborhood around it is unbounded. It also can't be an essential singularity by hypothesis.

(3) Find the singularities of $\frac{1}{\sin(z)}$. Show that they are poles. Give the residue and the order of each pole.

Solution: We know that $\sin(z) = 0 \Leftrightarrow z = \pi k,\ k \in \mathbb{Z}$. These are the singularities of $\frac{1}{\sin(z)}$. Moreover, $\sin(z)'|_{z=\pi k} = \cos(\pi k) = (-1)^k \neq 0$. So the singularities are simple poles. We have by L'Hôpital's rule:

$$\operatorname{res}_{\pi k} \frac{1}{\sin(z)} = \lim_{z \to \pi k} \frac{z}{\sin(z)} = \lim_{z \to \pi k} \frac{1}{\cos(z)} = (-1)^k.$$

(4) Find the singularities and the principal parts of the following function:
- (a) $(1 + z^4)^{-1}$,
- (b) $\frac{1-\cos(z)}{z^2}$,
- (c) $\frac{\sin(z)^2}{z^5}$,
- (d) $(1 + z^2)^{-2}$.

Solution:
- (a) The singularities are given by $z^4 = -1$, that is $z_k = e^{\pi i/4 + \pi i k/2}$, $k = 0, 1, 2, 3$. Each singularity is a simple pole, so the principal part is given by the residue. Note that

$1+z^4=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$. Then

$$\lim_{z\to z_k}\frac{z-z_k}{1+z^4}=\lim_{z\to z_k}\frac{1}{\prod_{\ell\neq k}(z-z_\ell)}=\frac{1}{\prod_{\ell\neq k}(z_k-z_\ell)}.$$

(b) The only singularity is at $z=0$. Note that $1-\cos(z)$ has a double zero at 0 since $(1-\cos(z))''=-\cos(z)$ which does not vanish for $z=0$. Hence we have a removable singularity at 0 and its principal part is 0.

(c) In this lecture, we defined

$$\sin(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}=z-\frac{z^3}{6}+\dots.$$

Dividing by z^5 shifts the series by 5 degrees. The principal part is then given by

$$\frac{1}{z^5}\left(z-\frac{z^3}{6}+\dots\right)^2=\frac{1}{z^3}-\frac{1}{3z}+\dots$$

(d) The singularities are given by $1+z^2=0$, that is $z=\pm i$. They are both poles of degree 2. We have

$$\frac{1}{(1+z^2)^2}=\frac{1}{(z-i)^2}\cdot\frac{1}{(z+i)^2}.$$

To compute the principal part, we only need to compute the Taylor series of $\frac{1}{(z\pm i)^2}$ at $z=\mp i$. The first coefficient is $\frac{1}{(\pm 2i)^2}=-\frac{1}{4}$. The second coefficient is $-\frac{2}{(\pm 2i)^3}=\pm\frac{i}{4}$. Then the principal part at $\mp i$ is given by

$$\frac{1}{(z\pm i)^2}\left(-\frac{1}{4}\pm\frac{i}{4}(z\pm i)\right)=-\frac{1}{4(z\pm i)^2}\pm\frac{i}{4(z-i)}.$$

(5) Let f be a holomorphic function with a simple pole at z_0 and g be a holomorphic function in a neighborhood of z_0. Show that

$$\operatorname{res}_{z_0}(fg)=g(z_0)\operatorname{res}_{z_0}f.$$

Solution: Since g is holomorphic at z_0, we have that $f(z)g(z)(z-z_0)$ is bounded in $\dot{D}(z_0,r)$ for some $r>0$. So fg has either a simple pole or a removable singularity at z_0. In the first case, the residue is

$$\lim_{z\to z_0}[f(z)g(z)(z-z_0)]=g(z_0)\lim_{z\to z_0}f(z)(z-z_0)=g(z_0)\operatorname{res}_{z_0}f.$$

In the second case, fg is bounded near z_0, so $f(z)g(z)(z-z_0)\to 0$ as $z\to z_0$. So $g(z_0)=0$ and

$$\operatorname{res}_{z_0}(fg)=0=g(z_0)\operatorname{res}_{z_0}f.$$

(6) Show that the Laurent series expansion is unique. That is, show that if

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\sum_{n=-\infty}^{\infty}b_n(z-z_0)^n$$

on some annulus centered at z_0, then $a_n=b_n$ for all n. *Hint: consider a contour integral around z_0 and use that the Laurent series converge uniformly in the annulus.*

Solution: Suppose that f and g converge on some annulus $A(z_0, R_1, R_2)$. Let m be fixed and γ be the circle $|z - z_0| = r$ with $R_1 < r < R_2$. By uniform convergence, we have

$$\int_\gamma f(z)(z-z_0)^{-m} dz = \int_\gamma \sum_{n=-\infty}^{\infty} a_n (z-z_0)^{n-m} dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-z_0)^{n-m} dz.$$

If $n - m \neq 1$, then the inner integral is 0 by Cauchy's theorem. Otherwise it was computed multiple times in the lecture. We get

$$\int_\gamma f(z)(z-z_0)^{-m} dz = 2\pi i a_{m+1}.$$

If $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$, then the above integral is $2\pi i b_{m+1}$. Therefore $a_{m+1} = b_{m+1}$ for all $m \in \mathbb{Z}$.

(7) Consider the following integrals from homework 3. Compute them using the residue theorem:

$$\int_{|z|=1} \left(\frac{1}{z} + e^z\right) dz, \qquad \int_{|z-1|=1} \frac{dz}{z^2-1}, \qquad \int_{|z|=1} \frac{e^z}{z^4} dz.$$

Solution: The first integral has a simple pole at 0 with residue 1. Therefore

$$\int_{|z|=1} \left(\frac{1}{z} + e^z\right) dz = 2\pi i.$$

The second integral has two simple poles at ± 1. Only $+1$ is inside the curve $|z-1| = 1$. The residue is

$$\lim_{z \to 1} \frac{z-1}{z^2-1} = \frac{1}{1+1} = \frac{1}{2}.$$

Then

$$\int_{|z-1|=1} \frac{dz}{z^2-1} = \frac{2\pi i}{2} = \pi i.$$

The third integral has only a pole at $z = 0$ with the following Laurent series:

$$\sum_{n=-4}^{\infty} \frac{z^n}{(n+4)!}.$$

In particular, the residue at 0 is $\frac{1}{3!} = \frac{1}{6}$. Therefore

$$\int_{|z|=1} \frac{e^z}{z^4} dz = \frac{2\pi i}{6} = \frac{\pi i}{3}.$$

(8) Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

by integrating the function $f(z) = \frac{1}{z^2+1}$ on the contour γ consisting of the real segment $[-R, R]$ and the upper semicircle going from R to $-R$ with center 0.

Solution: Let $f(z) = \frac{1}{z^2+1}$. We consider the contour given in the hint and write γ_R for the semicircle from R to $-R$. The singularities of f are $\pm i$. Only $+i$ is inside the contour. We have

$$\operatorname{res}_i f(z) = \lim_{z \to i} \frac{z-i}{z^2+1} = \frac{1}{2i}$$

By the residue theorem, we have

$$\int_{-R}^{R} f(z) dz + \int_{\gamma_R} f(z) dz = 2\pi i \operatorname{res}_i f = \pi.$$

Since $\left|z^2+1\right| \geq |z|^2 - 1$, we have $|f(z)| \leq \frac{1}{|z|^2-1}$ for z large enough. Then

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \frac{\pi R}{R^2-1} = O(R^{-1}).$$

Taking the limit as $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+1} = \pi.$$