SOLUTION 6

(1) Find the Laurent series of $\frac{1}{(z-1)(z-2)}$ in the annuli $A(0,0,1)$, $A(0,1,2)$ and $A(0,2,\infty)$. *Hint: use partial fractions and write each term as a geometric series.*

Solution: We have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

We want to apply $\frac{1}{1-x} = 1 + x + x^2 + \dots$ for $|x| < 1$. Suppose $|z| < 1$, then

$$\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} + \frac{1}{1-z} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}(1-2^{-n-1})z^n.$$

Similarly, if $1 < |z| < 2$, then

$$\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} - \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{\infty} 2^{-n-1} z^n - \sum_{n=1}^{\infty} z^{-n}.$$

Finally, if $2 < |z|$, then

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} - \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \sum_{n=1}^{\infty}(2^{n-1}-1)z^{-n} = \sum_{n=0}^{\infty}(2^{n-1}-1)z^{-n}.$$

(2) Use any method you like to find ($n > 0$ is an integer)

(a)

$$\int_{|z+1|=2} \frac{e^z}{(z+1)^{34}} dz,$$

(b)

$$\int_{|z-1|=1} \left(\frac{z}{z-1}\right)^n dz.$$

Solution:

(a) The integrand has a pole of order 34 at $z = -1$ and no other singularity. We compute its residue:

$$\operatorname{res}_{-1} \frac{e^z}{(z+1)^{34}} = \frac{1}{33!} \lim_{z\to -1} \frac{d^{33}}{dz^{33}} \frac{e^z}{(z+1)^{34}}(z+1)^{34} = \frac{1}{33!} \lim_{z\to -1} \frac{d^{33}}{dz^{33}} e^z = \frac{1}{33!} \lim_{z\to -1} e^z = \frac{1}{33!e}.$$

Then the integral is

$$\int_{|z+1|=2} \frac{e^z}{(z+1)^{34}} dz = 2\pi i \operatorname{res}_{-1} \frac{e^z}{(z+1)^{34}} = \frac{2\pi i}{33!e}.$$

(b) By Cauchy's formula for the function $f(z) = z^n$, we have

$$\int_{|z-1|=1} \left(\frac{z}{z-1}\right)^n dz = \int_{|z-1|=1} \frac{f(z)}{(z-1)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(1) = \frac{2\pi i}{(n-1)!} n! = 2\pi i n.$$

(3) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that $f(1/z)$ has a pole at $z = 0$ if and only if f is a non-constant polynomial. *Hint: start from the Taylor series of $f(z)$ at the origin.*

Solution: Since f is an entire function, it has a Taylor series converging on $\mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then $f(1/z) = \sum_{n=0}^{\infty} a_n z^{-n}$ for $z \in \mathbb{C}\backslash\{0\}$. By uniqueness of Laurent series, $f(1/z)$ has a pole if and only if there is $N > 0$ such that $a_n = 0$ for $n \geq N$. In that case, f is a polynomial of degree $N - 1$.

(4) Prove the identity

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}$$

for $z \in \mathbb{C}\backslash\mathbb{Z}$. *Bonus:* deduce that $\zeta(2) = \frac{\pi^2}{6}$. *Hint: show that the difference of the two sides extends to a bounded entire function whose limit as* $\mathrm{Im}(z) \to \pm\infty$ *is 0.*

Solution: Let $f(z) = (\frac{\pi}{\sin(\pi z)})^2$ and $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}$. First note that $g(z)$ converges uniformly away from $\mathbb{Z}$: remove the two integers closest to z from the sum. Then the two next integer closest to z are at distance at least one. The two next are at distance at least 2 and so on. In the end, we bound g by $2\sum_{n=1}^{\infty} \frac{1}{n^2}$ plus finitely many terms.

The two functions have poles of order 2 at integers. We compute the principal part of $f(z)$. First we compute the Laurent series of $\frac{\pi}{\sin(\pi z)}$ at $z_0 = n \in \mathbb{Z}$. We have

$$\lim_{z\to n} \frac{\pi(z-n)}{\sin(\pi z)} = \frac{\pi}{\pi\cos(\pi z)} = (-1)^n$$

and, using the Taylor series of $\sin(\pi z)$ and $\cos(\pi z)$ at $z = n$

$$\begin{aligned}
\lim_{z\to n} \frac{d}{dz}\frac{\pi(z-n)}{\sin(\pi z)} &= \lim_{z\to n} \frac{\pi\sin(\pi z) - \pi^2(z-n)\cos(\pi z)}{\sin(\pi z)^2} \\
&= \lim_{z\to n} \frac{(-1)^n\pi^2(z-n) - (-1)^2\pi^2(z-n) + O((z-n)^3)}{\sin(\pi z)^2} \\
&= \lim_{z\to n} \frac{O((z-n)^3)}{\sin(\pi z)^2} \\
&= 0.
\end{aligned}$$

This limit can also be computed using L'Hôpital. Therefore $\frac{\pi}{\sin(\pi z)} = \frac{(-1)^n}{z-n} + 0 + O(z-n)$. We conclude that

$$\left(\frac{\pi}{(\sin(\pi z)}\right)^2 = \left(\frac{(-1)^n}{z-n} + 0 + O((z-n))\right)^2 = \frac{1}{(z-n)^2} + O(1).$$

So f and g have the same principal part. Then $f - g$ has removable singularities at $n \in \mathbb{Z}$. We want to show that $f - g$ is bounded. By periodicity, we can suppose that $\mathrm{Re}(z) \in [0, 1]$. We have

$$|\sin(\pi z)| = \frac{\left|e^{\pi iz} - e^{-\pi iz}\right|}{2} \geq \frac{\left|e^{-\pi\,\mathrm{Im}(z)} - e^{\pi\,\mathrm{Im}(z)}\right|}{2} \to \infty$$

as $\mathrm{Im}(z) \to \infty$. In particular, $|f(z)| < 1$ if $|\mathrm{Im}(z)| > R$ for some R large enough. Similarly for g, suppose $R < |\mathrm{Im}\,z|$. Note that $|z + n|$ is always larger than the absolute value of its real or imaginary part. Then

$$|g(z)| \leq \sum_{|n|\leq R} \frac{1}{|z+n|^2} + \sum_{|n|>R} \frac{1}{|z+n|^2} \leq \frac{2R+1}{R^2} + \sum_{|n|>R} \frac{1}{(\mathrm{Re}(z)+n)^2} = O(R^{-1}).$$

The second estimate is obtained by comparing the sum to the integral $\int_R^\infty \frac{dt}{\mathrm{Re}(z)+t}$. As for f, we have $|g(z)| \leq 1$ for R large enough.

The remaining part of the strip, $\{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,1] \ \ |\mathrm{Im}(z)| \leq R\}$, is a compact set. On it $f - g$ have only removable singularities so it is bounded. In conclusion, $f - g$ is a bounded function on $\mathbb{C}$. By Liouville's theorem, it is a constant. Since f and g goes to 0 as $\mathrm{Im}(z) \to \infty$, the constant is 0 and $f = g$.

Finally

$$2\zeta(2) = \lim_{z\to 0}\left(g(z) - \frac{1}{z^2}\right) = \lim_{z\to 0}\left(f(z) - \frac{1}{z^2}\right).$$

We have

$$\frac{\pi^2}{\sin(\pi z)^2} - \frac{1}{z^2} = \frac{\pi^2 z^2 - \sin(\pi z)^2}{\sin(\pi z)^2 z^2} = \frac{\pi^2 z^2 - \left[(\pi z)^2 - (\pi z)^4/3 + O(z^6)\right]}{\sin(\pi z)^2 z^2} = \frac{\pi^2(\pi z)^2/3 + O(z^4)}{\sin(\pi z)^2}.$$

Using again $\lim_{z\to 0} \frac{\pi z}{\sin(\pi z)} = 1$, we get

$$2\zeta(2) = \frac{\pi^2}{3}\lim_{z\to 0}\frac{(\pi z)^2}{\sin(\pi z)^2} = \frac{\pi^2}{3}.$$

(5) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot \pi.$$

Hint: follow the method of exercise (8) in homework 5.

Solution: Like in homework 5, we consider the contour given by the segment $[-R, R]$ and the semicircle γ_R going from R to $-R$. We have

$$\left|\frac{1}{(1+x^2)^{n+1}}\right| \leq \frac{1}{|R^2-1|^{n+1}} \to 0$$

as $R \to \infty$. Then the residue theorem tells us that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = 2\pi i \,\mathrm{res}_i \frac{1}{(1+z^2)^{n+1}}.$$

Clearly, the function $f(z) = \frac{1}{(1+z^2)^{n+1}}$ has a pole of order $n+1$ at i. We compute

$$\begin{aligned}
\mathrm{res}_i f &= \frac{1}{n!}\lim_{z\to i}\frac{d^n}{dz^n} f(z)(z-i)^n \\
&= \frac{1}{n!}\lim_{z\to i}\frac{d^n}{dz^n}\frac{1}{(z+i)^{n+1}} \\
&= \frac{1}{n!}\lim_{z\to i}(-1)^n\frac{(n+1)(n+2)\cdots(2n)}{(z+i)^{2n+1}} \\
&= \frac{(n+1)(n+2)\dots(2n)}{2^{2n+1}n!i} \\
&= \frac{1}{2i}\frac{(2n)!}{2^{2n}n!n!} \\
&= \frac{1}{2i}\frac{(2n)!}{[2\cdot 4\cdots(2n-2)(2n)]^2} \\
&= \frac{1}{2i}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots(2n-2)(2n)}.
\end{aligned}$$

In conclusion, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{2\pi i}{2i}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots(2n-2)(2n)} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots(2n-2)(2n)}\pi.$$

(6) Construct a function f that has a non-isolated singularity and countably many singularities. *Hint: consider the z such that $e^{1/z}$ is 1.*

Solution: We have $e^{1/z} = 1 \Leftrightarrow \frac{1}{z} \in 2\pi i\mathbb{Z} \Leftrightarrow z = \frac{1}{2\pi ik}$ for some $k \in \mathbb{Z}$. Then as $k \to \infty$, $\frac{1}{2\pi ik} \to 0$. So the function $\frac{1}{e^{1/z}-1}$ has a non-isolated singularity at 0. The singularities of that functions are exactly $\frac{1}{2\pi ik}$ for $k \in \mathbb{Z}$ and 0.

(7) Let $f(z) = \frac{(z-2)^3 e^z}{(z-1)^4}$. Find

$$\int_{|z|=r} \frac{f'(z)}{f(z)} dz$$

for $r = 3$ and $r = 3/2$.

Solution: The curve $|z| = r$ is simple. By the argument principle, we need to count the zeros and poles of $f(z)$ inside the curve. Clearly f has a zero of order 3 at 2, a pole of order 4 at 1 and no other singularity. Therefore

$$\int_{|z|=r} \frac{f'(z)}{f(z)} dz = 2\pi i \cdot \begin{cases} -1 & \text{if } r = 3, \\ -4 & \text{if } r = 3/2. \end{cases}$$

(8) Let $f(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial. Show that if $|a_k|\, r^k > \sum_{j \neq k} |a_j|\, r^j$ for some $r > 0$, then f has exactly k roots inside $D(0, r)$. *Hint: use Rouché's theorem.*

Solution: Let $g(z) = a_k z^k$ and $h(z) = \sum_{j \neq k} a_j z^j$. Then on $|z| = r$, we have $|g(z)| \geq |h(z)|$. By Rouché's theorem, $g(z)$ and $g(z) + h(z) = f(z)$ have the same number of zeros inside $D(0, r)$. So f has exactly k roots inside $D(0, r)$.